第2章 静电场中的物质与电场能量

§2.1 静电场中的导体

§2.2 电容与电容器

§2.3 静电场中的介质

§2.4 静电场的能量

§ 2.3.3 电介质的基本电学特性

1. 高斯定理

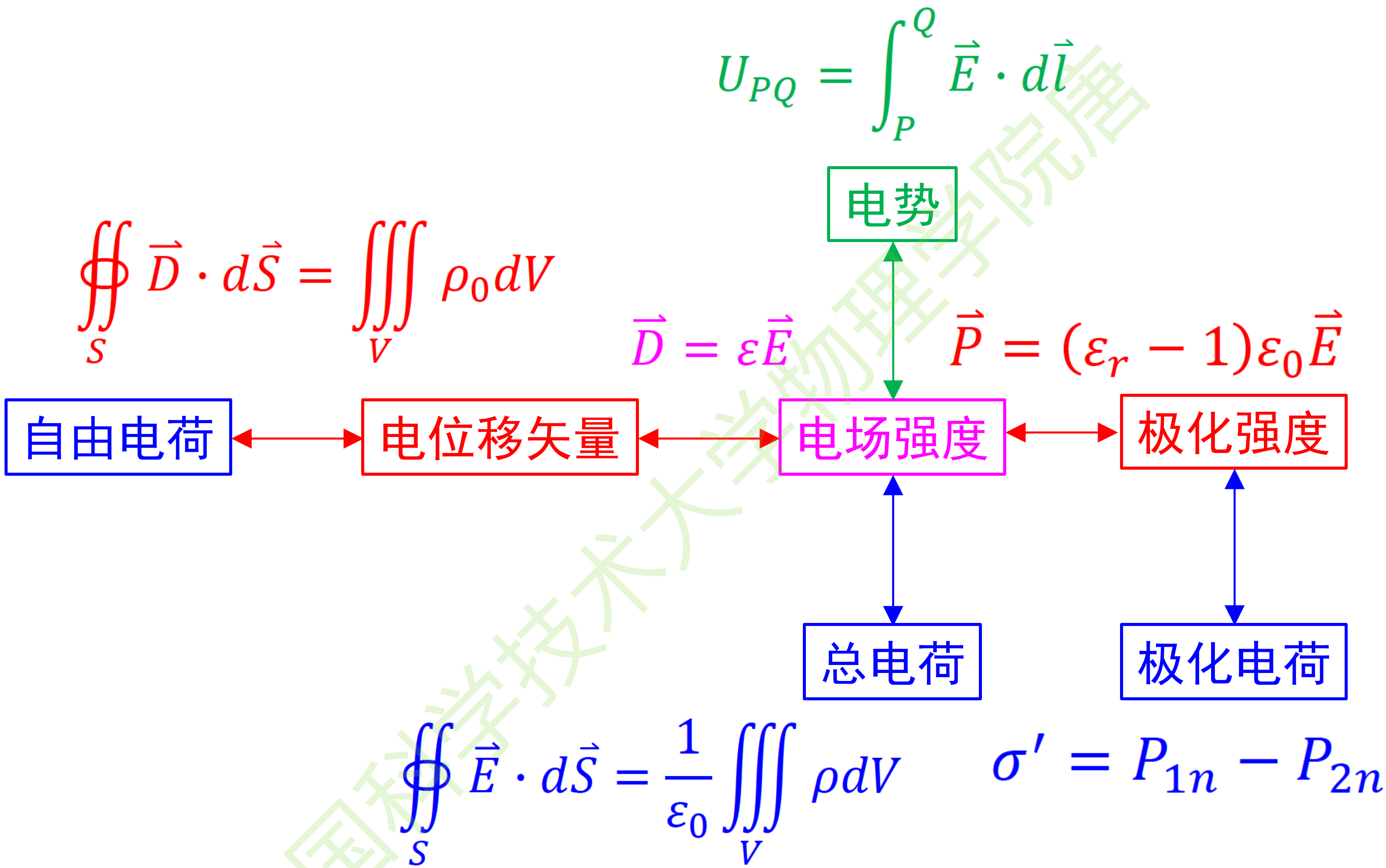
$$\oiint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_V \rho dV$$

$$\oiint_S \varepsilon_0\vec{E}\cdot d\vec{S} = \iiint_V \rho dV = \iiint_V \rho_0 dV + \iiint_V \rho' dV$$

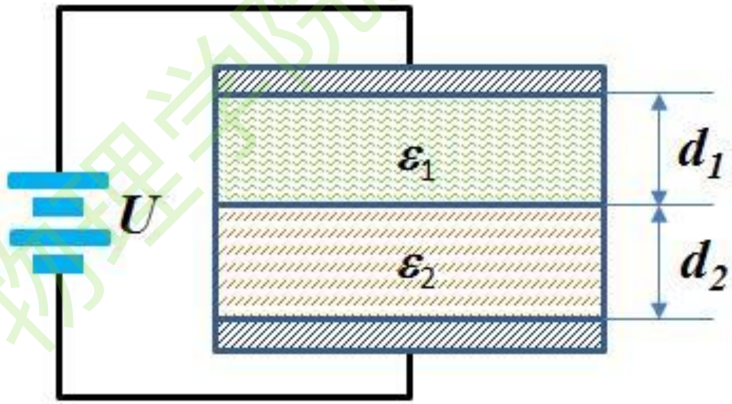
$$\iiint_V \rho' dV = -\oiint_S \vec{P}\cdot d\vec{S}$$

$$\oiint_S (\varepsilon_0\vec{E}+\vec{P})\cdot d\vec{S} = \iiint_V \rho_0 dV$$

$$\boxed{\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_0 dV}$$


$U_{PQ} = \int_P^Q \vec{E} \cdot d\vec{l}$
电势
$\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho_0 dV$
$\vec{D} = \varepsilon \vec{E}$
$\vec{P} = (\varepsilon_r - 1)\varepsilon_0 \vec{E}$
自由电荷
电位移矢量
电场强度
极化强度
总电荷
极化电荷
$\oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho dV$
$\sigma' = P_{1n} - P_{2n}$

【例 2.4】平行板电容器内充满两层均匀电介质，电容器所加电压为U。求：（1）电容器的电容；（2）介质表面上的极化电荷和总电荷密度。



【解】设极板上自由电荷密度为$\pm\sigma_0$

则两种电介质中的电位移矢量为：

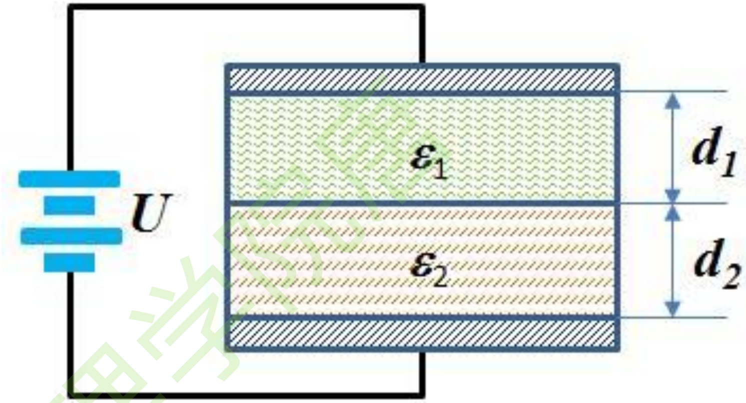
$$D_1 = D_2 = \sigma_0$$

可得电场强度为：

$$E_1 = \frac{D_1}{\varepsilon_1} = \frac{\sigma_0}{\varepsilon_1} \qquad E_2 = \frac{D_2}{\varepsilon_2} = \frac{\sigma_0}{\varepsilon_2}$$

$$\boxed{\frac{E_1}{E_2} = \frac{\varepsilon_2}{\varepsilon_1}}$$

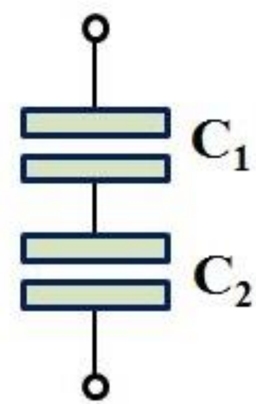
电势差为：



$$U=E_1d_1+E_2d_2=\sigma_0\left(\frac{d_1}{\varepsilon_1}+\frac{d_2}{\varepsilon_2}\right)$$

电容为：
$$C=\frac{Q_0}{U}=\frac{\sigma_0 S}{\sigma_0\left(\frac{d_1}{\varepsilon_1}+\frac{d_2}{\varepsilon_2}\right)}=\frac{\varepsilon_0 S}{\left(\frac{d_1}{\varepsilon_{r1}}+\frac{d_2}{\varepsilon_{r2}}\right)}$$

两个电容串联：



$$C_1=\frac{\varepsilon_{r1}\varepsilon_0 S}{d_1}\qquad C_2=\frac{\varepsilon_{r2}\varepsilon_0 S}{d_2}$$

$$C=\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}=\frac{\varepsilon_0 S}{\left(\frac{d_1}{\varepsilon_{r1}}+\frac{d_2}{\varepsilon_{r2}}\right)}$$

$$E_1 = \frac{D_1}{\varepsilon_1} = \frac{\sigma_0}{\varepsilon_1} \qquad E_2 = \frac{D_2}{\varepsilon_2} = \frac{\sigma_0}{\varepsilon_2}$$

$$U = E_1 d_1 + E_2 d_2 = \sigma_0 \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right)$$

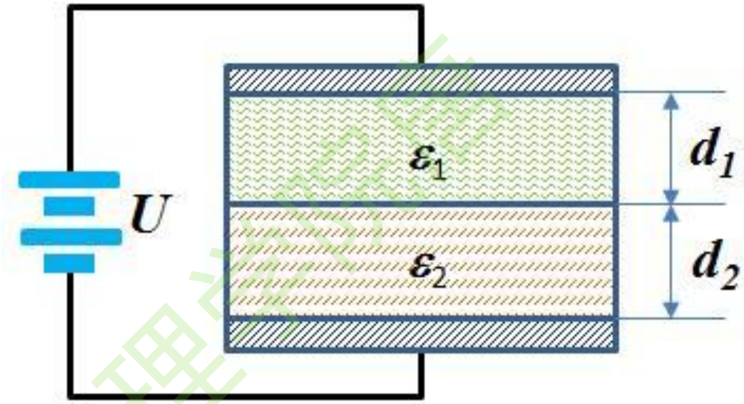
$$\sigma_0 = \frac{U}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} = \frac{\varepsilon_0 U}{\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}}} > \sigma_0^0 = \varepsilon_0 E_0^0 = \frac{\varepsilon_0 U}{d_1 + d_2}$$

$$E_1 = \frac{1}{\varepsilon_1} \frac{U}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} = \frac{1}{\varepsilon_{r1}} \frac{U}{\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}}}$$

$$E_2 = \frac{1}{\varepsilon_2} \frac{U}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} = \frac{1}{\varepsilon_{r2}} \frac{U}{\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}}}$$

$$E_0^0 = \frac{U}{d_1 + d_2}$$

上极板与电介质1交界面处的电荷：



- 自由电荷密度：

$$\sigma_0 = CU/S$$

- 极化电荷密度：

$$\sigma' = -P_1 = -(\varepsilon_1 - \varepsilon_0)E_1 = -(\varepsilon_1 - \varepsilon_0)\frac{\sigma_0}{\varepsilon_1} = -\left(1 - \frac{1}{\varepsilon_{r1}}\right)\sigma_0$$

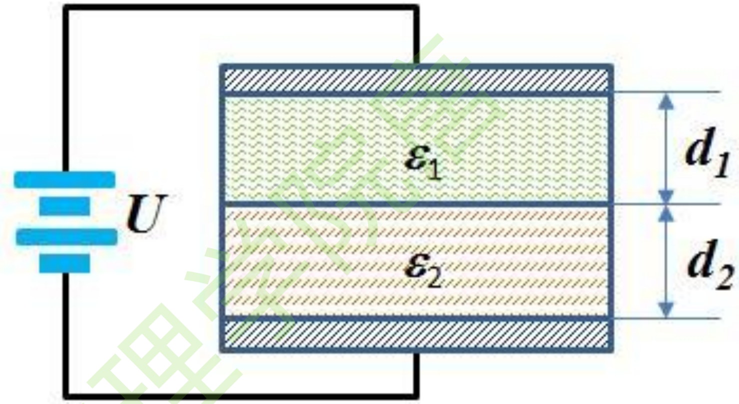
- 总电荷密度：

$$\sigma = \varepsilon_0 E_1 = \frac{\varepsilon_0 \sigma_0}{\varepsilon_1} = \frac{\sigma_0}{\varepsilon_{r1}}$$

$$\sigma = \sigma_0 + \sigma' = \frac{\sigma_0}{\varepsilon_{r1}}$$

总电荷密度小于自由电荷密度

下极板与电介质2交界面处的电荷：



- 自由电荷密度：

$$-\sigma_0 = -CU/S$$

- 极化电荷密度：

$$\sigma' = P_2 = (\varepsilon_2 - \varepsilon_0)E_2 = (\varepsilon_2 - \varepsilon_0)\frac{\sigma_0}{\varepsilon_2} = \left(1 - \frac{1}{\varepsilon_{r2}}\right)\sigma_0$$

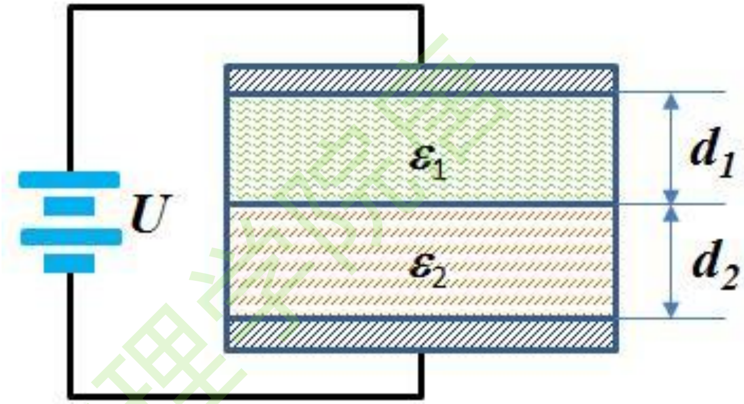
- 总电荷密度：

$$\sigma = -\varepsilon_0 E_2 = -\frac{\varepsilon_0\sigma_0}{\varepsilon_2} = -\frac{\sigma_0}{\varepsilon_{r2}}$$

$$\sigma = -\sigma_0 + \sigma' = -\frac{\sigma_0}{\varepsilon_{r2}}$$

上极板总电荷密度不等于下极板

电介质1与电介质2交界面处的电荷：



- 自由电荷密度：

$$D_2 - D_1 = 0 \qquad \boxed{D_1 = D_2}$$

- 极化电荷密度：

$$\sigma' = -(P_2 - P_1) = -\left[\left(1 - \frac{1}{\varepsilon_{r2}}\right)\sigma_0 - \left(1 - \frac{1}{\varepsilon_{r1}}\right)\sigma_0\right]$$

$$= \left(\frac{1}{\varepsilon_{r2}} - \frac{1}{\varepsilon_{r1}}\right)\sigma_0$$

- 总电荷密度：

$$\sigma = \varepsilon_0(E_2 - E_1) = \varepsilon_0\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\sigma_0 = \left(\frac{1}{\varepsilon_{r2}} - \frac{1}{\varepsilon_{r1}}\right)\sigma_0 \neq 0$$

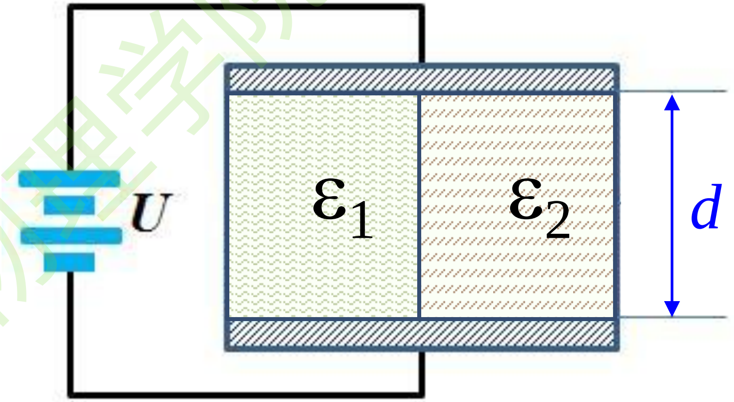
2. 环路定理

$$\boxed{\oint_L \vec{E}\cdot d\vec{l}=0}$$

$$\boxed{\nabla\times\vec{E}=0}$$

- 外电场具有无旋性
- 极化电荷产生的电场同样具有无旋性
- 总电场仍具有无旋性，满足环路定理

【例】平行板电容器内充满俩列均匀电介质，电容器所加电压为U。求：（1）电容器的电容；（2）介质表面上的极化电荷和总电荷密度。



【解】两种电介质中的电场强度为：

$$E_1 = E_2 = \frac{U}{d}$$

$$\boxed{E_1 = E_2}$$

则两种电介质中的电位移矢量为：

$$D_1 = \varepsilon_1 E_1 = \frac{\varepsilon_1 U}{d}$$

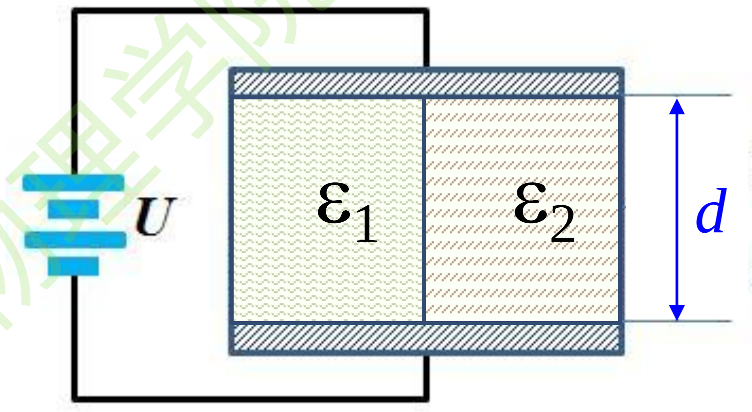
$$D_2 = \frac{\varepsilon_2 U}{d}$$

$$\boxed{\frac{D_1}{D_2} = \frac{\varepsilon_1}{\varepsilon_2}}$$

极板与两种电介质交界面上的自由电荷密度：

$$\sigma_{01} = D_1 = \frac{\varepsilon_1 U}{d}$$

$$\sigma_{02} = D_2 = \frac{\varepsilon_2 U}{d}$$

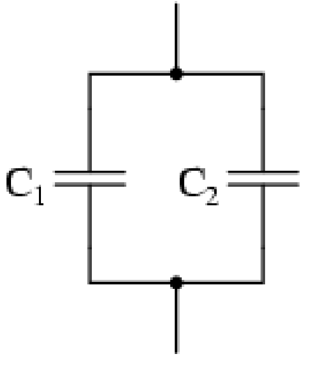


电容为：

$$C = \frac{Q}{U} = \frac{(\sigma_{01} + \sigma_{02})S/2}{U} = \frac{(\varepsilon_1 + \varepsilon_2)S}{2d}$$

两个电容并联：

$$C = C_1 + C_2 = \frac{(\varepsilon_1 + \varepsilon_2)S}{2d}$$



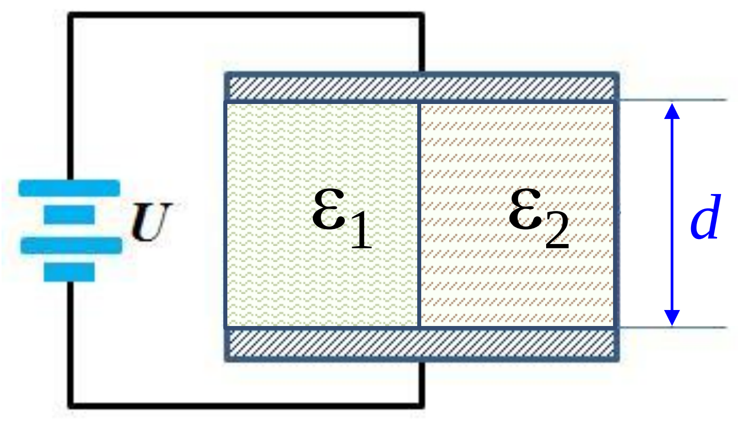
极板与两种电介质交界面上的自由电荷密度：

$$\sigma_{01} = D_1 = \frac{\varepsilon_1 U}{d} \qquad \sigma_{02} = D_2 = \frac{\varepsilon_2 U}{d}$$

极化电荷密度：

$$\sigma_1' = -P_1 = -(\varepsilon_1 - \varepsilon_0)E_1 = -(\varepsilon_1 - \varepsilon_0)\frac{U}{d}$$

$$\sigma_2' = -P_2 = -(\varepsilon_2 - \varepsilon_0)E_2 = -(\varepsilon_2 - \varepsilon_0)\frac{U}{d}$$

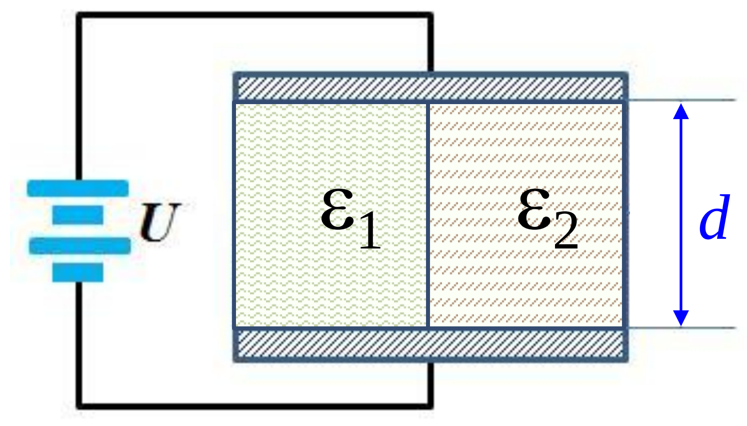


总电荷密度：

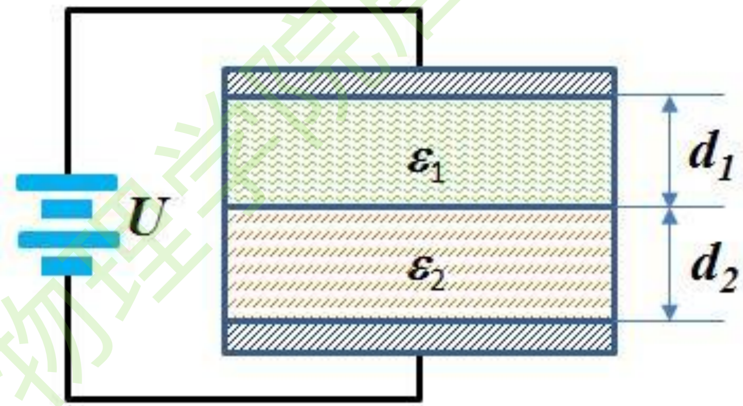
$$\sigma_1 = \sigma_{01} + \sigma_1' = \frac{\varepsilon_0 U}{d} \qquad \sigma_2 = \sigma_{02} + \sigma_2' = \frac{\varepsilon_0 U}{d}$$

$$\sigma_1 = \varepsilon_0 E_1 = \frac{\varepsilon_0 U}{d} = \varepsilon_0 E_2 = \sigma_2$$

- 自由电荷密度不同，极化电荷不同
- 总电荷密度相同



电介质交界面的边值关系



- 法线方向

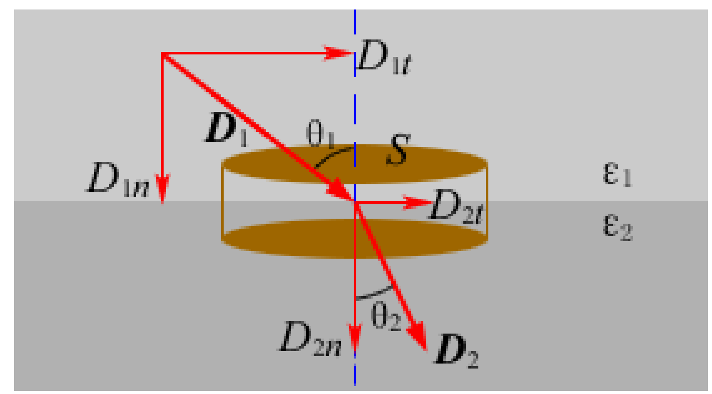
$$\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_0 dV$$

$$\boxed{D_{2n} - D_{1n} = \sigma_0}$$

一般情况下，电介质表面无自由电荷

$$\boxed{D_{2n} = D_{1n}}$$

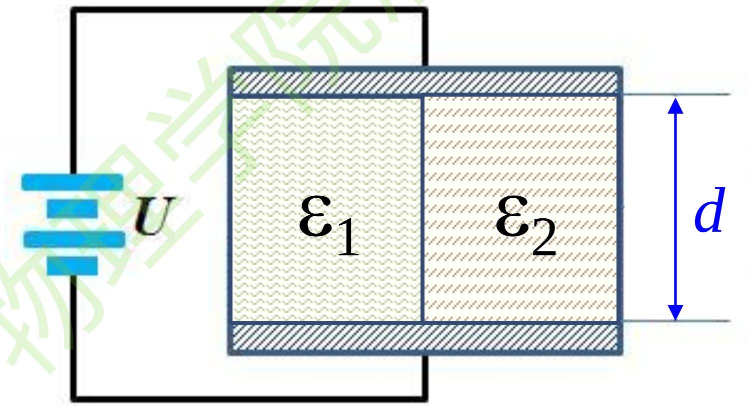
$$\boxed{\frac{E_{2n}}{E_{1n}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$



电介质交界面的边值关系

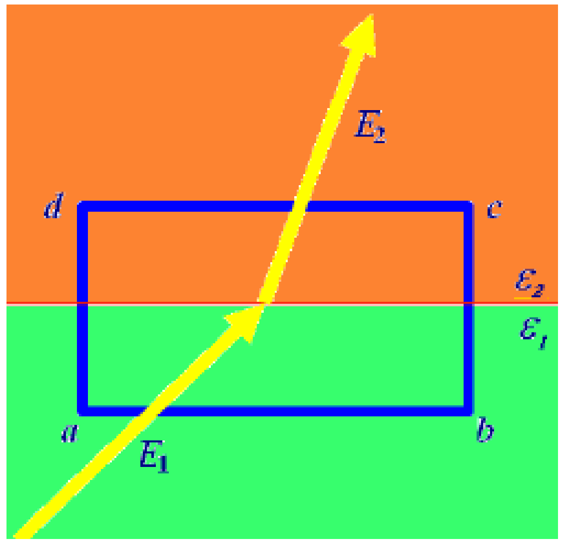
- 切线方向

$$\oint_L \vec{E} \cdot d\vec{l} = 0$$



$$\boxed{E_{2t} = E_{1t}}$$

$$\boxed{\frac{D_{2t}}{D_{1t}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$



介质中电学问题的基本方程

介质中的电学规律

$$\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_0 dV$$

$$\oint_L \vec{E}\cdot d\vec{l} = 0$$

介质中的本构方程

$$\vec{D} = \varepsilon\vec{E}$$

介质中的边值关系

$$D_{2n} = D_{1n}$$

$$E_{2t} = E_{1t}$$

$$U_{PQ} = \int_P^Q \vec{E} \cdot d\vec{l}$$

$$\vec{E} = -\nabla U$$

$$\sigma_0 = D_{2n} - D_{1n}$$

$$\oint\!\!\!\oint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho_0 dV$$

电势

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{P} = (\varepsilon_r - 1)\varepsilon_0 \vec{E}$$

自由电荷 ⟷ 电位移矢量 ⟷ 电场强度 ⟷ 极化强度

总电荷

极化电荷

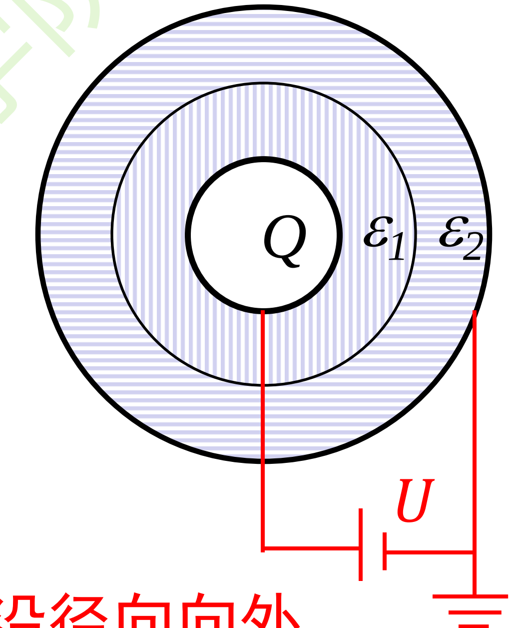
$$\oint\!\!\!\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho dV$$

$$\iiint_V \rho' dV = -\oint\!\!\!\oint_S \vec{P} \cdot d\vec{S}$$

$$\sigma = \varepsilon_0 (E_{2n} - E_{1n})$$

$$\sigma' = P_{1n} - P_{2n}$$

【例】一导体球与导体球壳同心，半径分别为R_1和R_3，中间充满两层介电常数分别为ε_1和ε_2的电介质，电介质交界面的半径为R_2。现将导体球和球壳分别与电动势为U的电压源相连，求介质中的电场分布。



【解】： 根据高斯定理：

$$D \cdot 4\pi r^2 = Q$$

$$D_1 = D_2 = D = \frac{Q}{4\pi r^2}$$

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon_1 r^2} & R_1 < r < R_2 \\ \dfrac{Q}{4\pi\varepsilon_2 r^2} & R_2 < r < R_3 \end{cases}$$

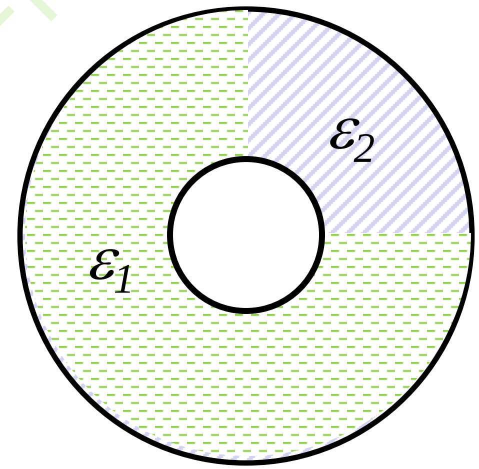
方向沿径向向外

$$U = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\varepsilon_{r1}}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{1}{\varepsilon_{r2}}\left(\frac{1}{R_2} - \frac{1}{R_3}\right)\right]$$

求得Q并代入，易得E

【例】一导体球与导体球壳同心，半径分别为R_1和R_2，中间充满两层介电常数分别为ε_1和ε_2的电介质。设导体球带自由电荷Q，求空间中的电场分布。

【解】：（1）根据高斯定理：



$$D \cdot 4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}$$

$$E = \begin{cases} 0 & r < R_1 \\ \dfrac{Q}{4\pi\varepsilon_1 r^2} & R_1 < r < R_2, 0 < \phi < \dfrac{3\pi}{2}, 0 < \theta < \pi \\ \dfrac{Q}{4\pi\varepsilon_2 r^2} & R_1 < r < R_2, \dfrac{3\pi}{2} < \phi < 2\pi, 0 < \theta < \dfrac{\pi}{2} \\ \dfrac{Q}{4\pi\varepsilon_0 r^2} & r > R_3 \end{cases}$$

【解】： 根据边值关系：

$$E_1(r) = E_2(r) = E(r)$$

$$D_1(r) = \varepsilon_1 E(r) \quad D_2(r) = \varepsilon_2 E(r)$$

$$D_1(r) \neq D_2(r)$$

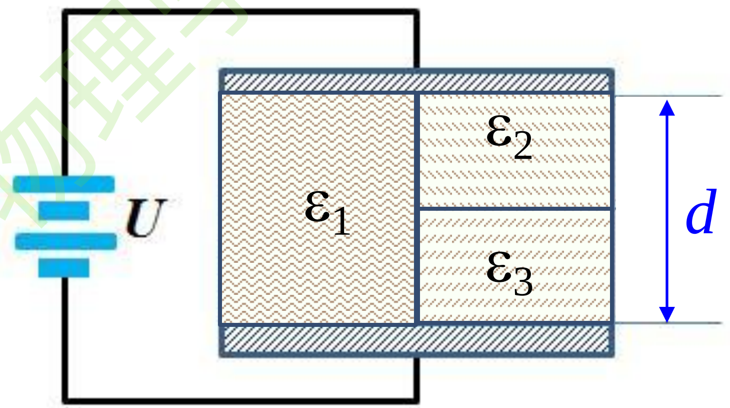
根据高斯定理：

$$D_1(r) \cdot \frac{7}{8} \cdot 4\pi r^2 + D_2(r) \cdot \frac{1}{8} \cdot 4\pi r^2 = Q$$

$$E(r) = \frac{1}{3.5\varepsilon_1 + 0.5\varepsilon_2} \frac{Q}{\pi r^2}$$

$$E(r) = \begin{cases} 0 & r < R_1 \\ \dfrac{1}{3.5\varepsilon_1 + 0.5\varepsilon_2} \dfrac{Q}{\pi r^2} & R_1 < r < R_2 \\ \dfrac{Q}{4\pi\varepsilon_0 r^2} & r > R_2 \end{cases}$$

【练习】平行板电容器内充满如图所示均匀电介质，左右介质水平面积相等，右边两介质厚度相等。电容器所加电压为U。求：（1）电容器中的电场分布；（2）电容器的电容；（3）介质表面上的极化电荷和总电荷密度。



§2.4 静电场的能量

§ 2.4.1 点电荷系统的静电相互作用能

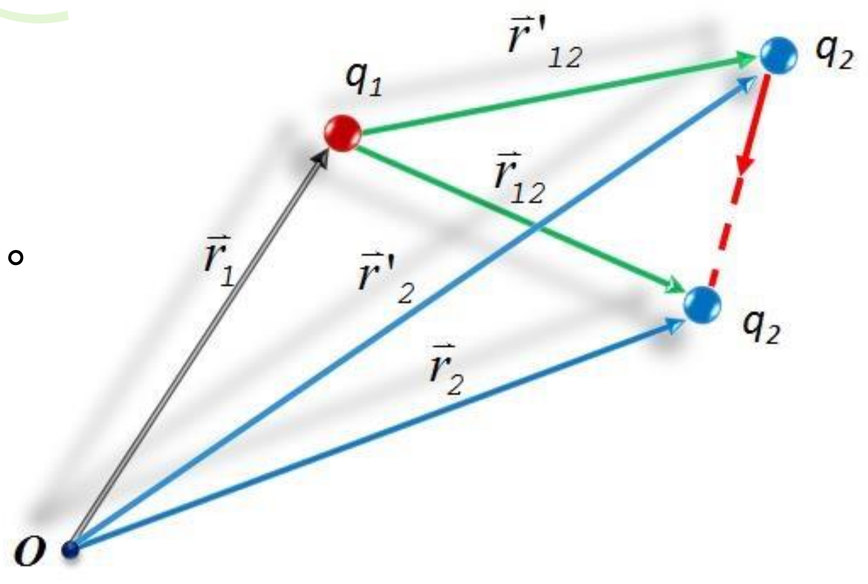
1. 两个点电荷系统

把q_1和q_2分别从无限远处移至r_1和r_2处。

先把q_1从无限远处移至r_1处，无需做功。

再把q_2从无限远处移至r_2处，

此过程需克服q_1的电场力做功。



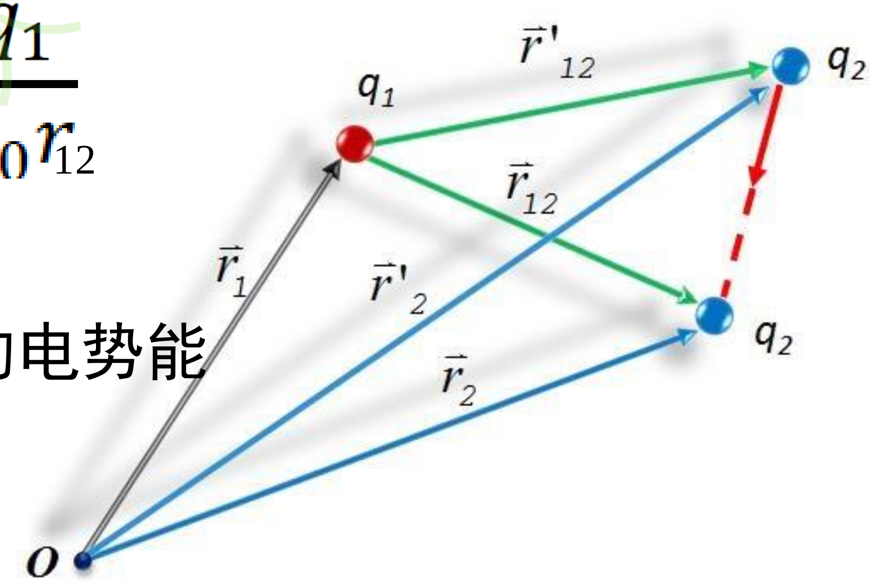
r_1处q_1在r_2处的电势为：

$$U_{12} = \frac{q_1}{4\pi\varepsilon_0 r_{12}}$$

将q_2从无限远移至r_2处，克服q_1的相互作用力做功为：

$$W_{12} = q_2 U_{12} = \frac{q_2 q_1}{4\pi\varepsilon_0 r_{12}}$$

也就是q_2在q_1产生的电场中的电势能



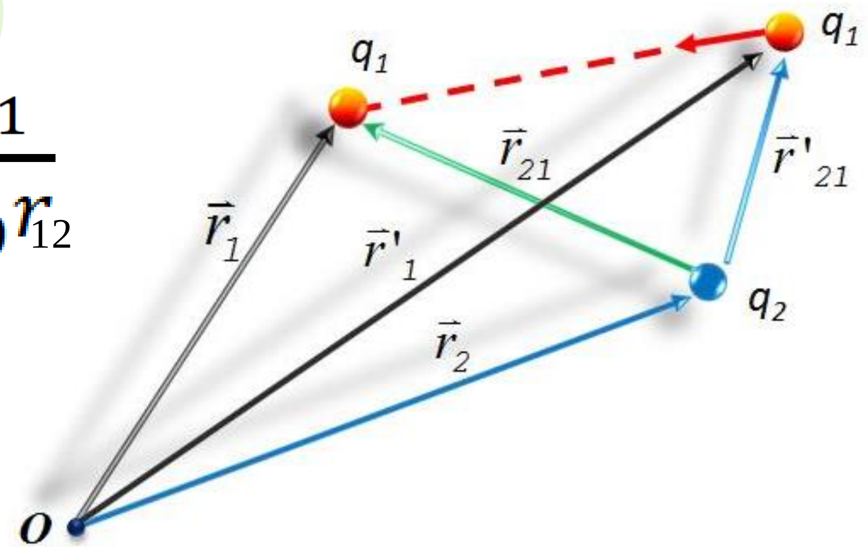
该过程的顺序也可以反过来，先移动q_2，再移动q_1

r_2处q_2在r_1处的电势为：

$$U_{21} = \frac{q_2}{4\pi\varepsilon_0 r_{21}}$$

将q_1从无限远移至r_1处，克服q_2的相互作用力做功为：

$$W_{21} = q_1 U_{21} = \frac{q_2 q_1}{4\pi\varepsilon_0 r_{12}}$$



$$\boxed{W_{21} = W_{12}}$$

做功的多少与移动顺序无关

相互作用能：建立这种电荷分布所需外界提供的能量

$$
\begin{aligned}
W_{互} &= W_{21} = q_1 U_{21} \\
&= W_{12} = q_2 U_{12} \\
&= \frac{1}{2}(q_1 U_{21} + q_2 U_{12}) \\
&= \frac{1}{2}\sum_{\mathrm{i}=1}^{2} q_i U_i
\end{aligned}
$$

U_i为q_i以外的电荷在r_i处产生的电势

系统中每个电荷的电势能之和除以2

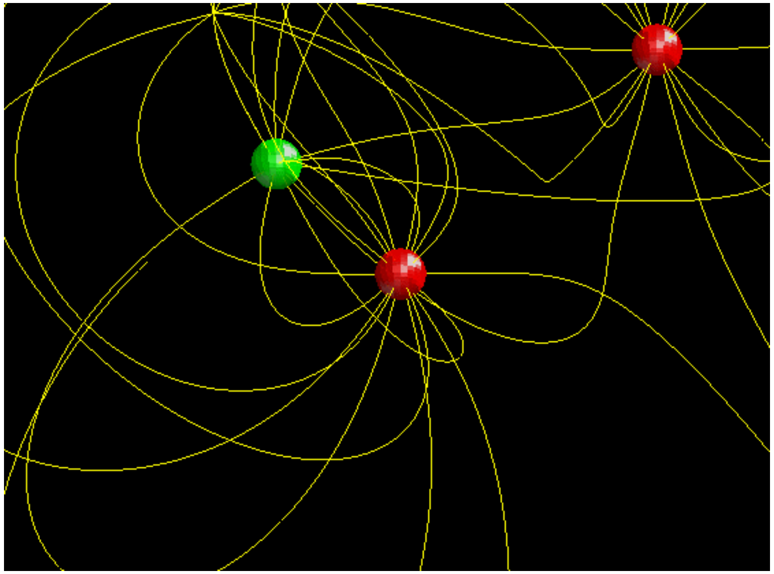
2. N个点电荷系统

将两个点电荷系统的相互作用能推广到N个点电荷系统

$$W_{互} = \frac{1}{2}\sum_{i=1}^{N} q_i U_i$$

U_i为q_i以外的电荷在r_i处产生的电势

$$U_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} U_{ji} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_j}{4\pi\varepsilon_0 r_{ij}}$$



$$W_{互} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_i q_j}{r_{ij}}$$