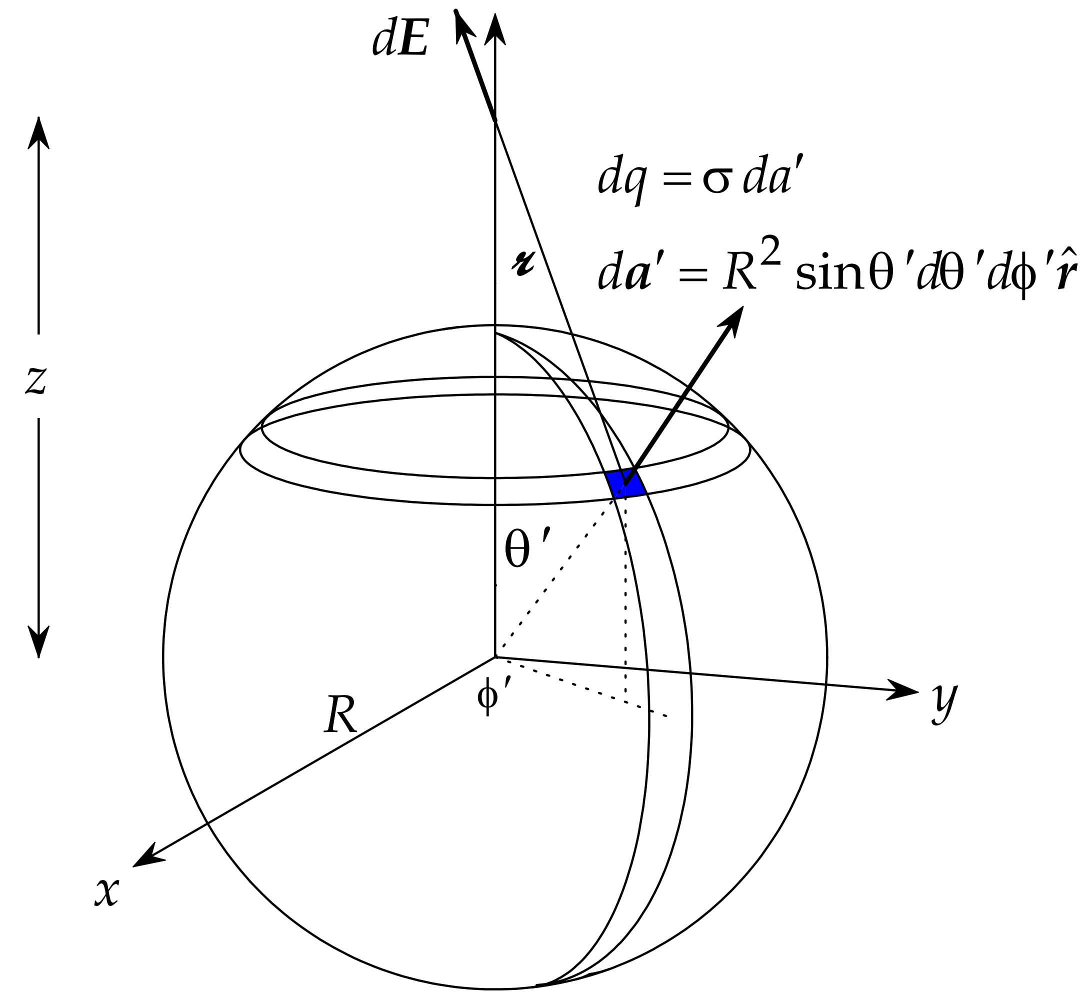
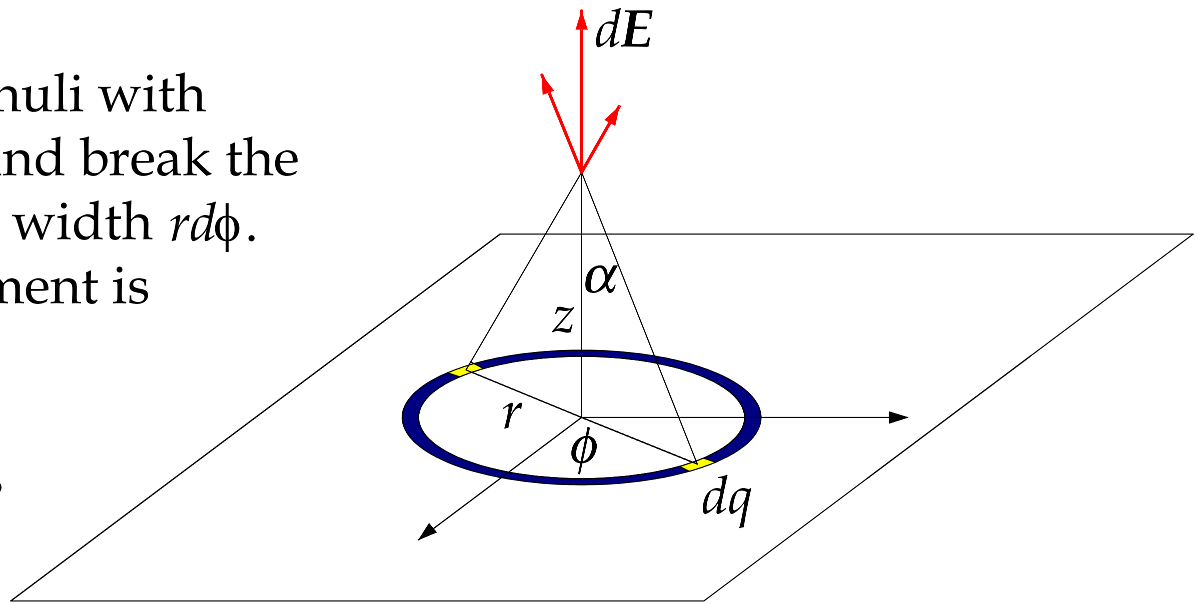
Today in Physics 217: charged spheres

- Finish example from Wednesday.
- Field from a charged spherical shell, calculated with Coulomb's Law.
- Same, calculated with Gauss' Law.
- Analogy with gravity: Gauss' Law for gravity.



Using Coulomb's Law:



Break the plane into annuli with radius r and width dr, and break the annuli into segments of width $rd\phi$. The charge of each segment is

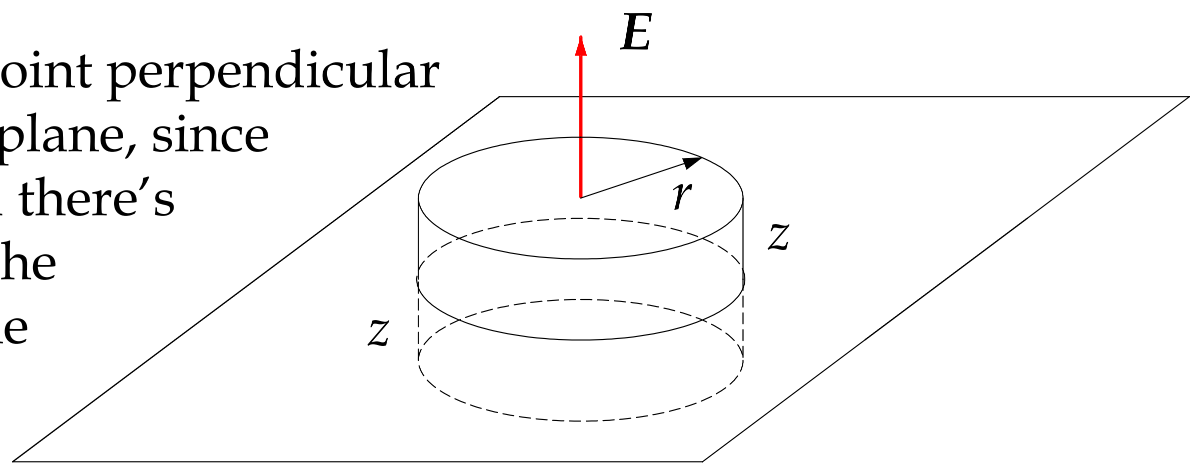
$$dq = \sigma\, r\, dr\, d\phi$$

Horizontal components of field from segments at ϕ and $\phi+\pi$ cancel, and their vertical components add, so above the plane, we have:

$$d\mathbf{E} = 2\frac{dq}{\mathscr{r}^2}\cos\alpha\,\hat{\mathbf{z}} = 2\frac{\sigma r}{z^2+r^2}\frac{z}{\sqrt{z^2+r^2}}\,dr\,d\phi\,\hat{\mathbf{z}}$$

$$E = 2\sigma z\hat{\mathbf{z}}\int_0^{\pi} d\phi \int_0^{\infty} r\left(r^2+z^2\right)^{-3/2} dr = \pi\sigma z\hat{\mathbf{z}}\int_{z^2}^{\infty} u^{-3/2}\,du = \pi\sigma z\hat{\mathbf{z}}\left.\frac{u^{-1/2}}{-1/2}\right|_{z^2}^{\infty} = 2\pi\sigma\,\hat{\mathbf{z}}$$

Using Gauss' Law



First note that $\boldsymbol{E}$ must point perpendicular to, and away from, the plane, since the plane is infinite and there's no difference between the view to the right and the view to the left. Then draw a cylinder, bisected by the plane. By symmetry, $\boldsymbol{E}$ is perpendicular to the area element vectors on the cylinder walls, parallel to those on the circular faces, and constant on those faces, so

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 2E\pi r^2 = 4\pi Q_{\text{enclosed}} = 4\pi^2 r^2 \sigma \text{, or}$$

$$\boldsymbol{E} = \pm 2\pi\sigma \hat{\boldsymbol{z}}.$$

Harder setup (finding and exploiting symmetry), easier math.

Electric fields from spherically-symmetrical charge distributions

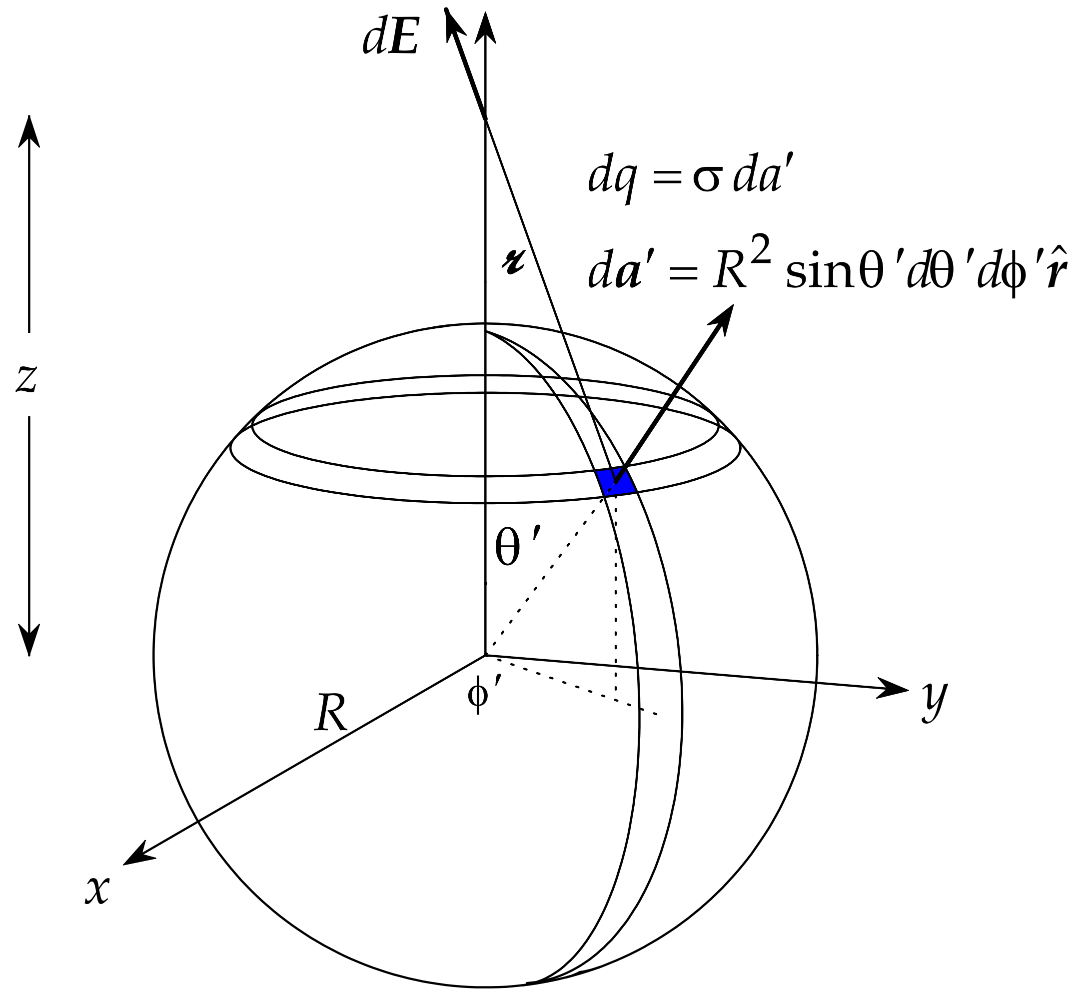
Today we will prove two important, though perhaps intuitively obvious, facts about spherical charge distributions:

- The field **outside** a uniformly-charged spherical shell is the same as that from a point charge of the same magnitude, the same distance away as the sphere's center.
- The field **inside** a uniformly-charged spherical shell is zero.

The proof will serve also as another useful example of the application of Coulomb's and Gauss' laws to the determination of electric fields from specified charge distributions.

Coulomb's Law example: field from a uniformly-charged spherical shell

Griffiths, problem 2.7: What is the electric field a distance *z* away from the center of a spherical shell with radius *R* and uniform surface charge density σ?



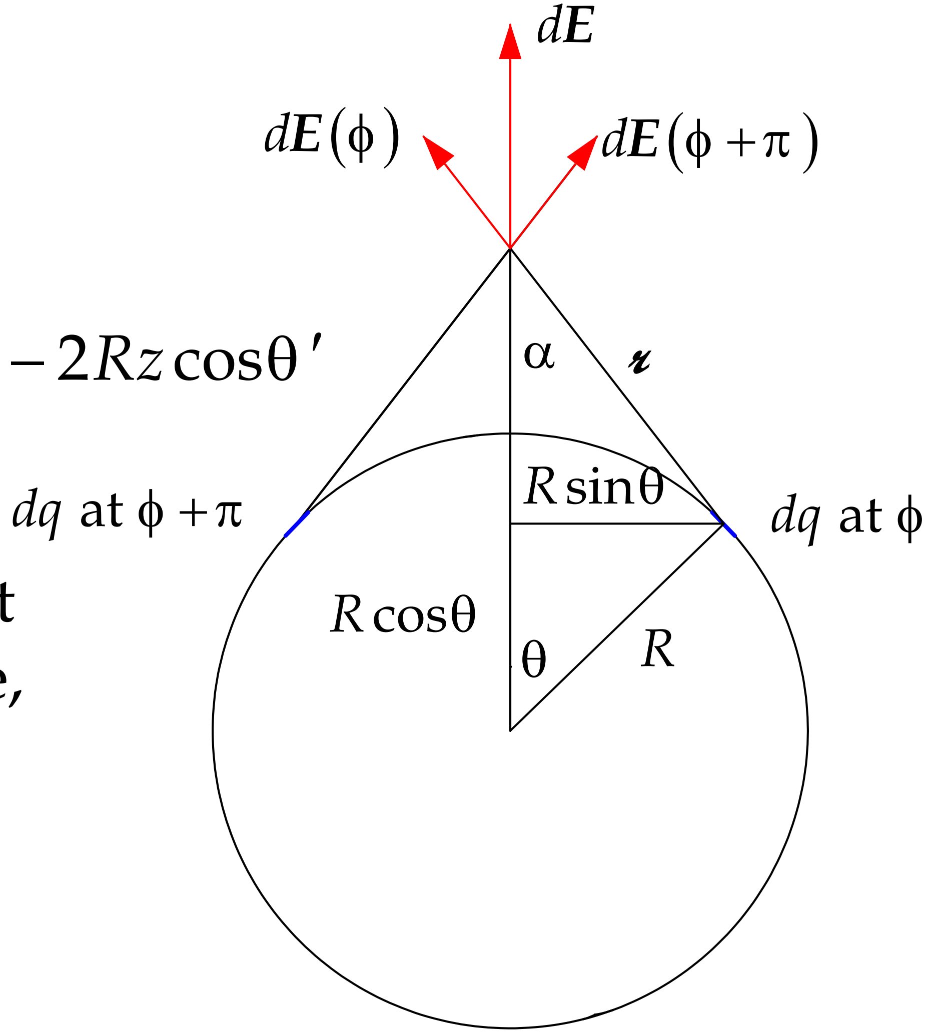
Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

In the plane at azimuth ϕ, it can be seen more easily that

$$\mathscr{r}^2 = R^2 \sin^2\theta' + (z - R\cos\theta')^2$$

$$= R^2 \sin^2\theta' + z^2 + R^2\cos^2\theta' - 2Rz\cos\theta'$$

$$= R^2 + z^2 - 2Rz\cos\theta'$$



Consider two area elements at azimuth ϕ and $\phi + \pi$: as before, the horizontal components of their contribution to E cancel, and the vertical components add.

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

So $$d\boldsymbol{E} = \hat{\boldsymbol{z}} 2 \frac{dq}{\boldsymbol{\mathscr{r}}^2} \cos\alpha$$

$$= \hat{\boldsymbol{z}} 2 \frac{\sigma R^2 \sin\theta' d\theta' d\phi'}{R^2 + z^2 - 2Rz\cos\theta'} \frac{z - R\cos\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= \hat{\boldsymbol{z}} 2\sigma R^2 \frac{\sin\theta' \left(z - R\cos\theta'\right)}{\left(R^2 + z^2 - 2Rz\cos\theta'\right)^{3/2}} d\theta' d\phi'$$

$$\boldsymbol{E} = \hat{\boldsymbol{z}} 2\sigma R^2 \int_0^\pi d\phi' \int_0^\pi \frac{\sin\theta' \left(z - R\cos\theta'\right)}{\left(R^2 + z^2 - 2Rz\cos\theta'\right)^{3/2}} d\theta'$$

The first integral is trivial: it just comes out to π.

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

For the second, substitute

$$w = \cos\theta', \quad dw = \cancel{-}\sin\theta' d\theta', \quad w = 1 \overset{\leftarrow}{\cancel{\rightarrow}} -1:$$

$$\mathbf{E} = \hat{\mathbf{z}} 2\pi\sigma R^2 \int_{-1}^{1} \frac{(z - Rw)}{\left(R^2 + z^2 - 2Rzw\right)^{3/2}} dw$$

Break this integral in two. For the first one, substitute

$$u = R^2 + z^2 - 2Rzw, \quad du = \cancel{-}2Rz\,dw,$$

$$u = R^2 + z^2 + 2Rz \overset{\leftarrow}{\cancel{\rightarrow}} R^2 + z^2 - 2Rz$$

$$z \int_{-1}^{1} \frac{1}{\left(R^2 + z^2 - 2Rzw\right)^{3/2}} dw = \frac{1}{2R} \int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-3/2} du$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\frac{1}{2R}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-3/2}du = \frac{1}{2R}\left[-2u^{-1/2}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$

$$= \frac{1}{R}\left[\frac{1}{\sqrt{R^2+z^2-2Rz}} - \frac{1}{\sqrt{R^2+z^2+2Rz}}\right]$$

The second half of the integral needs to be done by parts. Take

$$u = w \qquad dv = \frac{-R\,dw}{\left(R^2+z^2-2Rzw\right)^{-3/2}}$$

$$du = dw \qquad v = \frac{1}{z}\frac{1}{\sqrt{R^2+z^2-2Rzw}}$$ (as we just saw)

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\int_C u\,dv = uv\Big|_C - \int_C v\,du$$

$$\int_{-1}^{1} \frac{-Rw}{\left(R^2+z^2-2Rzw\right)^{3/2}}\,dw = \frac{1}{z}\frac{w}{\sqrt{R^2+z^2-2Rzw}}\Bigg|_{-1}^{1}$$

$$-\frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}}$$

In the last term, use (again)

$$u = R^2+z^2-2Rzw, \quad du = \cancel{-}2Rz\,dw,$$

$$u = R^2+z^2+2Rz \overset{\longleftarrow}{\cancel{\longrightarrow}} R^2+z^2-2Rz$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

and it becomes

$$
-\frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}} = -\frac{1}{2Rz^2}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2}\,du
$$

$$
= -\frac{1}{2Rz^2}\Big[2\sqrt{u}\,\Big]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}
$$

$$
= -\frac{1}{Rz^2}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)
$$

So, putting all these terms together (and factoring out $1/z^2$ as we do), we get

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\boldsymbol{E} = \hat{\boldsymbol{z}}\,\frac{2\pi\sigma R^2}{z^2}\left[\frac{z^2}{R}\left(\frac{1}{\sqrt{R^2+z^2-2Rz}}-\frac{1}{\sqrt{R^2+z^2+2Rz}}\right)\right.$$

$$+z\left(\frac{1}{\sqrt{R^2+z^2-2Rz}}+\frac{1}{\sqrt{R^2+z^2+2Rz}}\right)$$

$$\left.-\frac{1}{R}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)\right]$$

This looks like a mess until you notice that

$$\sqrt{R^2+z^2+2Rz} = \sqrt{(z+R)^2} = |z+R|$$

$$\sqrt{R^2+z^2-2Rz} = \sqrt{(z-R)^2} = |z-R|$$

Positive, since they represent the length of $\boldsymbol{\mathscr{r}}$, which is always positive.

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$
\begin{aligned}
\mathbf{E} = \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\Bigg[& \frac{z^2}{R}\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right) \\
& + z\left(\frac{1}{|z-R|}+\frac{1}{|z+R|}\right)-\frac{1}{R}\big(|z+R|-|z-R|\big)\Bigg] \\
= \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\Bigg[& \frac{z^2}{R}\left(\cancel{\frac{1}{|z-R|}}-\cancel{\frac{1}{|z+R|}}\right) \\
& + z\left(\frac{1}{|z-R|}+\frac{1}{|z+R|}\right)-\frac{1}{R}\left(\frac{\left|\cancel{z^2}-R^2\right|}{|z-R|}-\frac{\left|\cancel{z^2}-R^2\right|}{|z+R|}\right)\Bigg]
\end{aligned}
$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

which gives us, finally,

$$\boldsymbol{E} = \hat{\boldsymbol{z}} \frac{2\pi\sigma R^2}{z^2} \left[\frac{z-R}{|z-R|} + \frac{z+R}{|z+R|} \right]$$

Two cases: z larger than, or smaller than, R. (P outside, inside)

- **Larger (outside)**:

$$\frac{z-R}{|z-R|} = 1 = \frac{z+R}{|z+R|} \quad \Rightarrow \quad \boxed{\boldsymbol{E} = \hat{\boldsymbol{z}} \frac{4\pi\sigma R^2}{z^2} = \hat{\boldsymbol{z}} \frac{Q}{z^2}}$$

Behaves like a point charge at the sphere's center.

- **Smaller (inside)**: means $|z-R| = R-z$, so

$$\frac{z-R}{R-z} + \frac{z+R}{z+R} = \frac{z^2 - R^2 + R^2 - z^2}{R^2 - z^2} = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{E} = 0}$$

Gauss' Law example: field from a uniformly-charged spherical shell (continued)

First note that the field must be spherically symmetric as well, and point radially outward or inward – that is, ***E*** is perpendicular to all sphere's centered at the same point as the charged sphere. So draw two Gaussian spheres, one inside and one outside:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}}$$

$$r > R:$$

$$(E)\left(4\pi r^2\right) = 4\pi\left(4\pi R^2 \sigma\right) = 4\pi Q \quad \Rightarrow \quad \boldsymbol{E} = \hat{\boldsymbol{r}}\frac{Q}{r^2}$$

$$r < R:$$

$$(E)\left(4\pi r^2\right) = 0 \quad \Rightarrow \quad \boldsymbol{E} = 0$$

Gauss' Law for gravity

Newton was the first to realize these results, in the context of the other $1/r^2$ force, gravity. He convinced himself by use of a proof similar to our Coulomb's law demonstration, Gauss still not having been born by then. We could have saved Newton a lot of trouble by pointing out the following.

The force of gravity on a mass M from a mass m is

$$\boldsymbol{F} = \hat{\boldsymbol{\mathfrak{r}}} GmM \big/ \mathfrak{r}^2$$

Gravitational forces superpose: the force on M from N charges is

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{Gm_1 M}{\mathfrak{r}_1^2}\hat{\boldsymbol{\mathfrak{r}}}_1 + \frac{Gm_2 M}{\mathfrak{r}_2^2}\hat{\boldsymbol{\mathfrak{r}}}_2 + \ldots = M\sum_{i=1}^{N} G\frac{m_i}{\mathfrak{r}_i^2}\hat{\boldsymbol{\mathfrak{r}}}_i \equiv M\boldsymbol{g}(\boldsymbol{r})$$

Gauss' Law for gravity (continued)

For a continuous distribution of mass (density $\rho(\boldsymbol{r})$), the gravitational field $\boldsymbol{g}$ is obtained by letting $N \to \infty$:

$$\boldsymbol{g}(\boldsymbol{r}) = G \int_V \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\boldsymbol{r}') \, d\tau'$$

Take the divergence of both sides, and carry out the resulting integral on the RHS, as we did on Wednesday, and we get

$$\boxed{\nabla \cdot \boldsymbol{g} = 4\pi G \rho(\boldsymbol{r})}$$

Now integrate this result over volume, and use the divergence theorem, as we also did on Wednesday :

$$\boxed{\oint \boldsymbol{g} \cdot d\boldsymbol{a} = 4\pi G M_{\text{enclosed}}}$$