Positivity of holomorphic vector bundles in terms of L^p-conditions of $\bar{\partial}$

Jiafu Ning

Central South University

joint work with

Fusheng Deng, Zhiwei Wang and Xiangyu Zhou

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Basics of Hermitian holomorphic vector bundles

Let (X,ω) be a complex manifold of dimension n, and (E,h) be a Hermitiann holomorphic vector bundle of rank r over X. Let $D = D' + \bar\partial$ be the Chern connection of (E,h), and $\Theta_{E,h} = [D',\bar\partial] = D'\bar\partial + \bar\partial D'$. Denote by $(e_1,\cdots,e_r)$ an orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1,\cdots,z_n)$, and

$$i\Theta_{E,h} = i \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge d\bar z_k \otimes e^*_\lambda \otimes e_\mu, \quad \bar c_{jk\lambda\mu} = c_{kj\mu\lambda}.$$

To $i\Theta_{E,h}$ corresponds a natural Hermitian form $\theta_{E,h}$ on $TX \otimes E$ defined by

$$\theta_{E,h}(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(x) u_{j\lambda}\bar u_{k\mu}, \qquad u \in T_xX \otimes E_x.$$

$\theta(\xi\otimes s, \xi\otimes s)$, $\xi\otimes s \in T_xX \otimes E_x$.

Bochner-Kodaira-Nakano identity:
$\Delta' = D'D'^* + D'^*D'$ and $\Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting on E-valued forms satisfy the identity

$$\Delta'' = \Delta' + [i\Theta_{E,h}, \Lambda_\omega].$$

Let $x_0 \in X$ and $(z_1, \cdots, z_n)$ be local coordinates centered at x_0, such that $(\partial/\partial z_1, \cdots, \partial/\partial z_n)$ is an orthonormal basis of TX at x_0. One can write

$$\omega = i\sum dz_j \wedge d\bar z_j + O(\|z\|),$$

and

$$i\Theta_{E,h}(x_0) = i\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar z_k \otimes e_\lambda^* \otimes e_\mu,$$

where $(e_1, \cdots, e_r)$ is an orthonormal basis of E_{x_0}. Let $u = \sum u_{K,\lambda} dz \wedge d\bar z_K \otimes e_\lambda \in \Lambda^{n,q} T_X^* \otimes E$, where $dz = dz_1 \wedge \cdots \wedge dz_n$. Then

$$\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle = \sum_{|S|=q-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{jS,\lambda} \bar u_{kS,\mu}.$$

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$$\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle = \sum_{|S|=q-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{jS,\lambda} \bar{u}_{kS,\mu}.$$

In particular, if $q = 1$, we get

$$\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j,\lambda} \bar{u}_{k,\mu}.$$

So (E, h) is Nakano positive (resp. semo-positive) if and only if the Hermitian operator $[i\Theta_{E,h}, \Lambda_\omega]$ is positive definite (resp. semi-positive definite) on $\Lambda^{n,1} T_X^* \otimes E$.
If (E, h) is Nakano semo-positive and (E, h_1) is Nakano positive, then

$$\langle [i\Theta_{E,h} + i\Theta_{E,h_1}, \Lambda_\omega]u, u\rangle \geq \langle [i\Theta_{E,h_1}, \Lambda_\omega]u, u\rangle$$

and

$$\langle [i\Theta_{E,h} + i\Theta_{E,h_1}, \Lambda_\omega]^{-1} u, u\rangle \leq \langle [i\Theta_{E,h_1}, \Lambda_\omega]^{-1} u, u\rangle.$$

Theorem (L^2-estimate Theorem, see Demailly)

Let (X, ω) be a complete Kähler manifold, with a Kähler metric which is not necessarily complete. Let (E, h) be a Hermitian vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta_{E,h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\Lambda^{n,q}T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ satisfying $\bar\partial g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ such that $\bar\partial f = g$ and

$$\int_X |f|^2 dV_\omega \leq \int_X \langle B^{-1}g, g\rangle dV_\omega.$$

Definition

Let (X,ω) be a Kähler manifold of dimension n, which admits a positive line bundle, (E,h) be a (singular) Hermitian vector bundle (maybe of infinite rank) over X, and $p>0$.

(1) (E,h) satisfies *the optimal L^p-estimate condition* if for any positive line bundle (A,h_A) on X, for any $f\in\mathcal{C}^\infty_c(X,\wedge^{n,1}T^*_X\otimes E\otimes A)$ with $\bar\partial f=0$, there is $u\in L^p(X,\wedge^{n,0}T^*_X\otimes E\otimes A)$, satisfying $\bar\partial u=f$ and $\int_X|u|^p_{h\otimes h_A}dV_\omega\leq\int_X\langle B^{-1}_{A,h_A}f,f\rangle^{\frac{p}{2}}dV_\omega$, where $B_{A,h_A}=[i\Theta_{A,h_A}\otimes Id_E,\Lambda_\omega]$.

(2) (E,h) satisfies *the multiple coarse L^p-estimate condition* if for any $m\geq1$, for any positive line bundle (A,h_A) on X, and for any $f\in\mathcal{C}^\infty_c(X,\wedge^{n,1}T^*_X\otimes E^{\otimes m}\otimes A)$ with $\bar\partial f=0$, there is $u\in L^p(X,\wedge^{n,0}T^*_X\otimes E^{\otimes m}\otimes A)$, satisfying $\bar\partial u=f$ and $\int_X|u|^p_{h^{\otimes m}\otimes h_A}dV_\omega\leq C_m\int_X\langle B^{-1}_{A,h_A}f,f\rangle^{\frac{p}{2}}dV_\omega$, where C_m are constants satisfying $\frac{1}{m}\log C_m\to0$ as $m\to\infty$.

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Theorem (1)

Let (X,ω) be a Kähler manifold of dimension n, which admits a positive Hermitian holomorphic line bundle, (E,h) be a smooth Hermitian vector bundle over X, and $\theta\in C^0(X,\Lambda^{1,1}T^*_X\otimes End(E))$ such that $\theta^*=\theta$. If for any $f\in\mathcal{C}^\infty_c(X,\wedge^{n,1}T^*_X\otimes E\otimes A)$ with $\bar\partial f=0$, and any positive Hermitian line bundle (A,h_A) on X with $i\Theta_{A,h_A}\otimes Id_E+\theta>0$ on supp$f$, there is $u\in L^2(X,\wedge^{n,0}T^*_X\otimes E\otimes A)$, satisfying $\bar\partial u=f$ and $\int_X|u|^2_{h\otimes h_A}dV_\omega\leq\int_X\langle B^{-1}_{h_A,\theta}f,f\rangle_{h\otimes h_A}dV_\omega$, provided that the right hand side is finite, where $B_{h_A,\theta}=[i\Theta_{A,h_A}\otimes Id_E+\theta,\Lambda_\omega]$, then $i\Theta_{E,h}\geq\theta$ in the sense of Nakano. On the other hand, if in addition X is assumed to have a complete Kähler metric, the above condition is also necessary for that $i\Theta_{E,h}\geq\theta$ in the sense of Nakano. In particular, if (E,h) satisfies the optimal L^2-estimate condition, then (E,h) is Nakano semi-positive.

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Theorem (2)

Let (X, ω) be a Kähler manifold, which admits a positive Hermitian holomorphic line bundle, and (E, h) be a holomorphic vector bundle over X with a continuous Hermitian metric h. If (E, h) satisfies the multiple coarse L^p-estimate condition for some $p > 1$, then (E, h) is Griffiths semi-positive.

The case that $p = 2$ and h is Hölder continuous for the above theorem was proved by G. Hosono and T. Inayama, by showing that the multiple coarse L^2-estimate condition implies the the multiple coarse L^2-extension condition.

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L^p extension

B. Berndtsson and M. Păun(2010) and Q. A. Guan and X. Y. Zhou(2015) proved the L^p extension theorem.

Theorem

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, L be a complex affine line in $\mathbb{C}^n$, and $\Omega \cap L \neq \emptyset$. For $0 < p \leq 2$, then for any $f \in A^p(\Omega \cap L)$, there is $F \in A^p(\Omega)$, such that $F|_{\Omega \cap L} = f$ and

$$\int_{\Omega} |F|^p \leq C \int_{\Omega \cap L} |f|^p,$$

where C is a constant depending only on $\operatorname{diam} \Omega$ and n.

Let $\pi : X \to B$ be a proper holomorphic submersion from a complex n-dimensional Kähler manifold (X, ω) onto a unit ball in $\mathbb{C}^m$. Let $(E, h = h_E)$ be a Hermitian holomorphic vector bundle over X, such that the curvature $i\Theta_{E,h_E} \geq 0$ in the sense of Nakano. Let $t_0 \in B$ be an arbitrarily fixed point. Then for every section $u \in H^0(X_{t_0}, K_{X_{t_0}} \otimes E|_{X_{t_0}})$, such that $\int_{X_{t_0}} |u|^2_{\omega,h} dV_{\omega_{X_{t_0}}} < +\infty$, there is a section $\widetilde{u} \in H^0(X, K_X \otimes E)$, such that $\widetilde{u}|_{X_{t_0}} = \widetilde{u} \wedge dt$, with the following L^2-estimate

$$\int_X |\widetilde{u}|^2_{\omega,h} dV_{X,\omega} \leq \mu(B) \int_{X_{t_0}} |u|^2_{\omega,h} dV_{\omega_{X_{t_0}}},$$

where $dt = dt_1 \wedge \cdots \wedge dt_m$, and $t = (t_1, \cdots, t_m)$ are the holomorphic coordintes on $\mathbb{C}^m$, and $\mu(B)$ is the volume of the unit ball in $\mathbb{C}^m$ with respect to the Lebesgue measure on $\mathbb{C}^m$.

Definition

Let (E, h) be a Hermitian vector bundle over a domain $D \subset \mathbb{C}^n$ with a singular Finsler metric h, and $p > 0$.

(1) (E, h) satisfies *the optimal L^p-extension condition* if for any $z \in D$, and $a \in E_z$ with $|a| = 1$, and any holomorphic cylinder P with $z + P \subset D$, there is $f \in H^0(z + P, E)$ such that $f(z) = a$ and $\frac{1}{\mu(P)} \int_{z+P} |f|^p \leq 1$, where $\mu(P)$ is the volume of P with respect to the Lebesgue measure. (Here by a holomorphic cylinder we mean a domain of the form $A(P_{r,s})$ for some $A \in U(n)$ and $r, s > 0$, with $P_{r,s} = \{(z_1, z_2, \cdots, z_n) : |z_1|^2 < r^2, |z_2|^2 + \cdots + |z_n|^2 < s^2\}$).

(2) (E, h) satisfies *the multiple coarse L^p-extension condition* if for any $z \in D$, and $a \in E_z$ with $|a| = 1$, and any $m \geq 1$, there is $f_m \in H^0(D, E^{\otimes m})$ such that $f_m(z) = a^{\otimes m}$ and satisfies the following estimate: $\int_D |f_m|^p \leq C_m$, where C_m are constants independent of z and satisfying $\lim_{m\to\infty} \frac{1}{m} \log C_m = 0$.

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Two theorems

The following theorems are first prove by F.S. Deng, Z.W. Wang, L.Y. Zhang and X.Y. Zhou.

Theorem (3)

Let E be a holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local holomorphic section s of E^*. If (E, h) satisfies the optimal L^p-extension condition for some $p > 0$, then (E, h) is Griffiths semi-positive.

Theorem (4)

Let E be a holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local holomorphic section s of E^*. If (E, h) satisfies the multiple coarse L^p-extension condition for some $p > 0$, then (E, h) is Griffiths semi-positive.

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Lemma

Let $D \subset \mathbb{C}^n$ be a domain, and ϕ be an upper-semicontinuous function on D. If for any $z_0 \in D$ and any holomorphic cylinder P with with $z_0 + P \subset\subset D$,

$$\phi(z_0) \leq \frac{1}{\mu(P)} \int_{z_0+P} \phi,$$

then ϕ is plurisubharmonic on D.

Proof of Theorem (3)

Let u be a holomorphic section of E^* over D. Let $z \in D$ and P be any holomorphic cylinder such that $z + P \subset D$. Take $a \in E_z$ such that $|a|_h = 1$ and $|u|_{h^*}(z) = |\langle u(z), a\rangle|$. Since (E, h) satisfies the optimal L^p-extension condition, there is a holomorphic section f of E on $z + P$, such that $f(z) = a$ and satisfies the estimate

$$\frac{1}{\mu(P)} \int_{z+P} |f|_h^p \leq 1.$$

Note that $|u|_{h^*} \geq |\langle u, f\rangle|/|f|_h$ on $z + P$, it follows that

$$\log |u|_{h^*} \geq \log |\langle u, f\rangle| - \log |f|_h.$$

Taking integration, we get that

$$
\begin{aligned}
& p\left(\frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*}\right) \\
& \geq p\left(\frac{1}{\mu(P)}\int_{z+P}\log|\langle u, f\rangle|\right) - \frac{1}{\mu(P)}\int_{z+P}\log|f|_h^p \\
& \geq p\left(\frac{1}{\mu(P)}\int_{z+P}\log|\langle u, f\rangle|\right) - \log\left(\frac{1}{\mu(P)}\int_{z+P}|f|_h^p\right) \\
& \geq p\log|\langle u(z), f(z)\rangle| \\
& = p\log|\langle u(z), a\rangle| = p\log|u(z)|_{h^*},
\end{aligned}
$$

where the second inequality follows from Jensen's inequality, and the third inequality follows from the fact that $\log|\langle u, f\rangle|$ is a plurisubharmonic function. Dividing by p, we obtain that

$$\log|u(z)|_{h^*} \leq \frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*}.$$

By the Lemma above, we see that $\log|u|_{h^*}$ is plurisubharmonic on D.

Let u be a holomorphic section of E^* over D. Then $u^{\otimes m} \in H^0(D, (E^*)^{\otimes m})$.
Let $z \in D$ and P be any holomorphic cylinder such that $z + P \subset D$. Take $a \in E_z$ such that $|a|_h = 1$ and $|u|_{h^*}(z) = |\langle u(z), a\rangle|$. Since (E, h) satisfies the multiple coarse L^p-extension condition, there is $f_m \in H^0(D, E^{\otimes m})$, such that $f_m(z) = a^{\otimes m}$ and satisfies the following estimate

$$\int_D |f_m|^p \leq C_m,$$

where C_m are constants independent of z and satisfy the growth condition $\frac{1}{m} \log C_m \to 0$ as $m \to \infty$. Since $|u^{\otimes m}|_{(h^*)^{\otimes m}} = |u|^m_{h^*} \geq \frac{|\langle u^{\otimes m}, f_m\rangle|}{|f_m|_{h^{\otimes m}}}$, we have that

$$m \log |u|_{h^*} \geq \log |\langle u^{\otimes m}, f_m\rangle| - \log |f_m|.$$

Taking integration, we get that

$$
\begin{aligned}
& m\left(\frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*}\right) \\
\geq & \frac{1}{\mu(P)}\int_{z+P}\log|\langle u^{\otimes m}, f_m\rangle| - \frac{1}{p}\left(\frac{1}{\mu(P)}\int_{z+P}\log|f_m|^p\right) \\
\geq & m\log|u(z)|_{h^*} - \frac{1}{p}\log\left(\frac{1}{\mu(P)}\int_{z+P}|f_m|^p\right) \\
\geq & m\log|u(z)|_{h^*} - \frac{1}{p}\log\left(\frac{1}{\mu(P)}\int_{D}|f_m|^p\right) \\
\geq & m\log|u(z)|_{h^*} - \frac{1}{p}\log(C_m/\mu(P)),
\end{aligned}
$$

Dividing by m in both sides, we obtain that

$$\frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*} \geq \log|u(z)|_{h^*} - \frac{1}{mp}\log(C_m/\mu(P)).$$

Letting $m\to\infty$, we get $\log|u(z)|_{h^*} \leq \frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*}$.

L^2性质与曲率正性

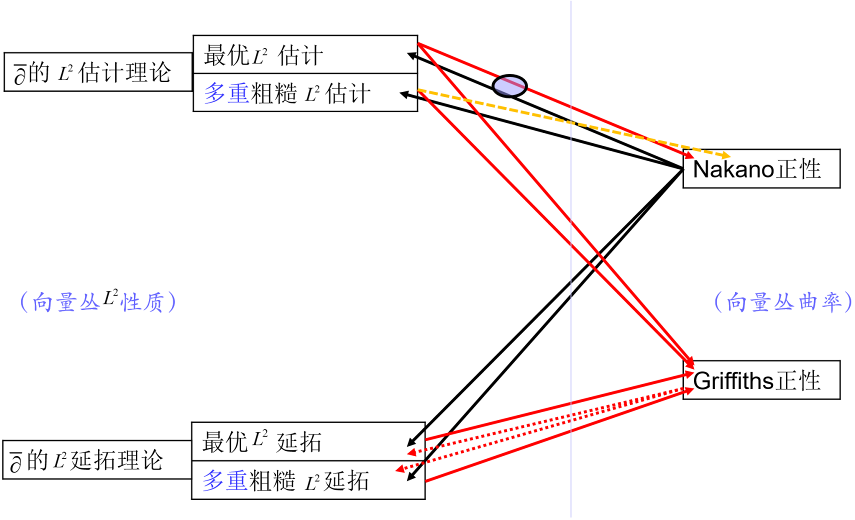

Figure 1: L^2

Some problems

Q1: Does Theorem (1) still hold if the optimal L^2-estimate condition is replaced by the optimal L^p-estimate condition for some $p \neq 2$?

Q2: Establish results analogous to Theorem (1) for holomorphic vector bundles with singular Hermitian metrics.

Q3: Does the multiple coarse L^2-estimate condition of (E, h) imply the Nakano positivity of (E, h)?

Q4: Does the Griffiths positivity imply the optimal L^2-extension condition and the multiple coarse L^2-extension condition?

Theorem (Berndtsson)

Let U and D be bounded domains in $\mathbb{C}^n_t$ and $\mathbb{C}^m_z$ respectively, and $\phi \in \mathcal{C}^2(\overline{U} \times \overline{D}) \cap PSH(U \times D)$. Assume that D is pseudoconvex. For $t \in U$, let $A^2_t := \{f \in \mathcal{O}(D) : ||f||^2_t := \int_D |f|^2 e^{-\phi(t,\cdot)} < \infty\}$ and $F := \coprod_{t \in U} A^2_t$. We may view F as a Hermitian holomorphic vector bundle on U. Then $(F, ||\cdot||_t)$ is Nakano semi-positive.

We will first prove that $(F, ||\cdot||_t)$ satisfies the $\bar\partial$ optimal L^2-estimate for pseudoconvex domains contained in U. We may assume U is pseudoconvex.
For any smooth strictly plurisubharmonic function ψ on U, for any $\bar\partial$ closed $f \in \mathcal{C}_c^\infty(T_U^*\Lambda^{(0,1)} \otimes F)$ (We identify $\mathcal{C}_c^\infty(T_U^*\Lambda^{(0,1)} \otimes F)$ with $\mathcal{C}_c^\infty(T_U^*\Lambda^{(n,1)} \otimes F)$). We may write $f = \sum_{j=1}^n f_j(t,z)d\bar t_j$ with $f_j(t,\cdot) \in F_t$ for $t \in U$ and $j = 1, 2, \cdots, n$. Therefore, we may view f as a $\bar\partial$-closed $(0,1)$-form on $U \times D$.

By L^2-estimate theorem, there exists a function u on $U \times D$, satisfying $\bar{\partial} u = f$ and

$$\begin{aligned}\int_{U\times D} |u|^2 e^{-(\phi+\psi)} &\leq \int_{U\times D} |f|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{-(\phi+\psi)} \\ &\leq \int_{U\times D} |f|^2_{i\partial\bar{\partial}\psi} e^{-(\phi+\psi)} \\ &= \int_U \sum_{j,k=1}^{n} \psi^{j\bar{k}} \langle f_j(t,\cdot), f_k(t,\cdot)\rangle_t e^{-\psi},\end{aligned}$$

where $(\psi^{j\bar{k}})_{n\times n} := (\frac{\partial^2\psi}{\partial t_j \partial_{\bar{t}_k}})^{-1}_{n\times n}$. Note that $\int_{U\times D} |u|^2 e^{-(\phi+\psi)} = \int_U ||u||_t^2 e^{-\psi} < \infty$ and $\frac{\partial u}{\partial \bar{z}_j} = 0$ for $j = 1, 2, \cdots, m$, we may view u as a L^2-section of F on U. By Theorem (1), $(F, ||\cdot||_t)$ is Nakano semi-positive.

Let $\Omega = U \times D \subset \mathbb{C}^n_t \times \mathbb{C}^m_z$ be a bounded pseudoconvex domains and $p : \Omega \to U$ be the natural projection. Let h be a Hermitian metric on the trivial bundle $E = \Omega \times \mathbb{C}^r$ that is C^2-smooth to $\overline{\Omega}$. For $t \in U$, let

$$F_t := \{f \in H^0(D, E|_{\{t\}\times D}) : \|f\|_t^2 := \int_D |f|^2_{h_t} < \infty\}$$

and $F := \coprod_{t \in U} F_t$. Since h is continuous to $\overline{\Omega}$, F_t are equal for all $t \in U$ as vector spaces. We may view $(F, \|\cdot\|)$ as a trivial holomorphic Hermitian vector bundle of infinite rank over U.

Theorem (direc image in stein)

Let θ be a continuous real $(1,1)$-form on U such that $i\Theta_E \geq p^*\theta \otimes Id_E$, then $i\Theta_F \geq \theta \otimes Id_F$ in the sense of Nakano. In particular, if $i\Theta_E > 0$ in the sense of Nakano, then $i\Theta_F > 0$ in the sense of Nakano.

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Proof

By Theorem (1), it suffices to prove that $(F, \|\cdot\|)$ satisfies: for any $f \in \mathcal{C}_c^\infty(U, \wedge^{n,1} T_U^* \otimes F \otimes A)$ with $\bar{\partial} f = 0$, and any positive Hermitian line bundle (A, h_A) on U with $i\Theta_{A,h_A} + \theta > 0$ on supp$f$, there is $u \in L^2(U, \wedge^{n,0} T_U^* \otimes F \otimes A)$, satisfying $\bar{\partial} u = f$ and

$$\int_U |u|^2_{h\otimes h_A} dV_\omega \leq \int_U \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A} dV_\omega,$$

provided that the right hand side is finite, where $B_{h_A,\theta} = [(i\Theta_{A,h_A} + \theta) \otimes Id_F, \Lambda_\omega]$.
We may write $f = \sum_{j=1}^n f_j(t,z) dt \wedge d\bar{t}_j$ with $f_j(t,\cdot) \in F_t \otimes A$ for $t \in U$ and $j = 1, 2, \cdots, n$. Therefore, we may view f as a $\bar{\partial}$-closed $E \otimes p^*A$-valued $(n,1)$-form on Ω. Let $\tilde{f} = f \wedge dz$, then $\tilde{f}$ is a $\bar{\partial}$-closed $E \otimes p^*A$-valued $(m+n,1)$-form on Ω. By assumption, $i\Theta_E \geq p^*\theta \otimes Id_E$.

We get

$$i\Theta_E + ip^*(\Theta_{A,h_A}) \otimes Id_E \geq p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E.$$

Therefore,

$$\begin{aligned} &\langle [i\Theta_E + ip^*(\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A} \\ \leq &\langle [p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A} \end{aligned}$$

By L^2-estimate theorem, we can find an $E \otimes p^*A$-valued $(n+m, 0)$-form $\tilde{u}$ on Ω, satisfying $\bar{\partial}\tilde{u} = \tilde{f}$ and

$$\begin{aligned} &\int_\Omega |\tilde{u}|^2_{h\otimes h_A} \\ \leq &\int_\Omega \langle [p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A} \\ = &\int_U \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A}, \end{aligned}$$

where the last equality holds by the Fubini theorem.

Since $\frac{\partial \tilde{u}}{\partial \bar{z}_j} = 0$, $\tilde{u}$ is holomorphic along fibers and we may view $u = \tilde{u}/dz$ as a section of $K_U \otimes F \otimes A$. Also by the Fubini theorem, we have

$$\int_\Omega |\tilde{u}|^2_{h\otimes h_A} = \int_U ||u||^2_{h\otimes h_A} < \infty.$$

We also have $\bar{\partial} u = f$. Hence $(F, \|\cdot\|)$ satisfies the optimal L^2-estimate condition and is Nakano semi-positive by Theorem (1).

Let $\pi : X \to U$ be a proper holomorphic submersion from Kähler manifold X of complex dimension $m+n$, to a bounded pseudoconvex domain $U \subset \mathbb{C}^n$, and (E,h) be a Hermitian holomorphic vector bundle over X, with the Chern curvature Nakano semi-positive. From L^2-extension theorem, the direct image $F := \pi_*(K_{X/U} \otimes E)$ is a vector bundle, whose fiber over $t \in U$ is $F_t = H^0(X_t, K_{X_t} \otimes E|_{X_t})$. There is a hermtian metric $\|\cdot\|$ on F induced by h: for any $u \in F_t$,

$$\|u(t)\|_t^2 := \int_{X_t} c_m u \wedge \bar{u},$$

where $m = \dim X_t$, $c_m = i^{m^2}$, and $u \wedge \bar{u}$ is the composition of the wedge product and the inner product on E. So we get a Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U.

Theorem

The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U defined above satisfies the optimal L^2-estimate condition. Moreover, if $i\Theta_E \geq p^*\theta \otimes Id_E$ for a continuous real $(1,1)$-form θ on U, then $i\Theta_F \geq \theta \otimes Id_F$ in the sense of Nakano.

Remark

The conclusion that $i\Theta_F \geq \theta \otimes Id_F$ is semipositive in the sense of Nakano is proved by Berndtsson, our proof is different with his.

Theorem

The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ *over* U *defined above satisfies the optimal* L^2*-estimate condition. Moreover, if* $i\Theta_E \geq p^*\theta \otimes Id_E$ *for a continuous real* $(1,1)$*-form* θ *on* U*, then* $i\Theta_F \geq \theta \otimes Id_F$ *in the sense of Nakano.*

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Theorem

The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U satisfies the multiple coarse L^2-estimate condition. In particular, F is Griffiths semipositive.

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Proof. For any $t_0 \in U$, any holomorphic cylinder P such that $t_0 + P \subset U$, and any $a_{t_0} \in F_{t_0}$, which is a holomorphic section of $K_{X_{t_0}} \otimes E|_{X_{t_0}}$ on X_{t_0}. Since E is Nakano semi-positive, from the optimal L^2-extention Theorem, we get a homolomorphic extension $a \in H^0(X, K_X \otimes E)$ such that $a|_{X_{t_0}} = a_{t_0} \wedge dt$, and with the estimate
$\int_{\pi^{-1}(t_0+P)} c_{m+n} a \wedge \bar{a} \leq \mu(P) \int_{X_{t_0}} c_m a_{t_0} \wedge \bar{a}_{t_0} = \mu(P)|a_{t_0}|^2_{t_0}$,
where $\mu(P)$ is the volume of P with respect to the Lebesgue measure $d\mu$ on $\mathbb{C}^m$. Since
$a_t := (a/dt)|_{X_t} \in H^0(X_t, K_{X_t} \otimes E|_{X_t})$, a/dt can be seen as a holomorphic section of the direct image bundle F over $t_0 + P$, and from Fubini's theorem, we can obtain that
$\int_{t_0+P} |a_t|^2_t dV_{\omega_0} \leq \mu(P)|a_{t_0}|^2_{t_0}$.

Theorem

The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U satisfies the optimal L^2-extension condition. In particular, F is Griffiths semipositive.

Proof. For any $t_0 \in U$, any holomorphic cylinder P such that $t_0 + P \subset U$, and any $a_{t_0} \in F_{t_0}$, which is a holomorphic section of $K_{X_{t_0}} \otimes E|_{X_{t_0}}$ on X_{t_0}. Since E is Nakano semi-positive, from the optimal L^2-extention Theorem, we get a homolomorphic extension $a \in H^0(X, K_X \otimes E)$ such that $a|_{X_{t_0}} = a_{t_0} \wedge dt$, and with the estimate

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Thank You!