Bounding the Distance between a Loop Subdivision Surface and Its Limit Mesh

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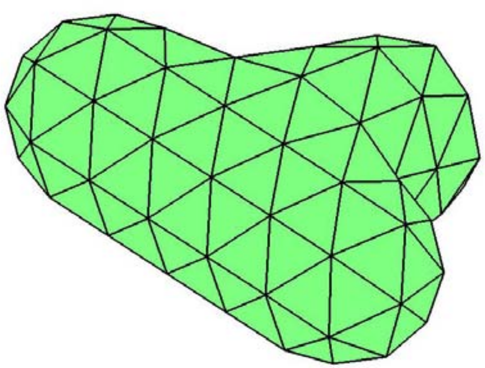
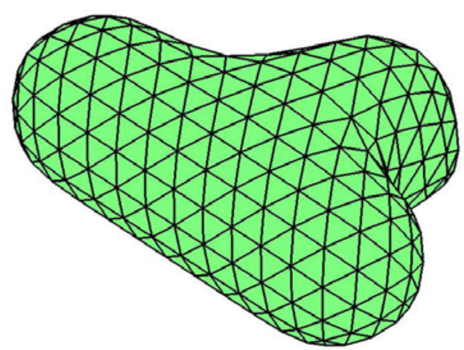
Loop subdivision surface

Loop subdivision surface (Loop 1987)

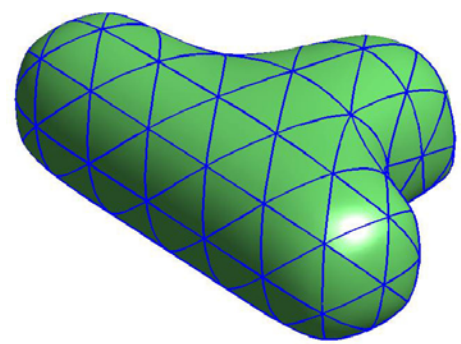
- a generalization of quartic three-directional box spline surface to triangular meshes of arbitrary topology
  - C^2 continuous except at extraordinary points
- the limit of a sequence of recursively refined (triangular) control meshes



initial mesh



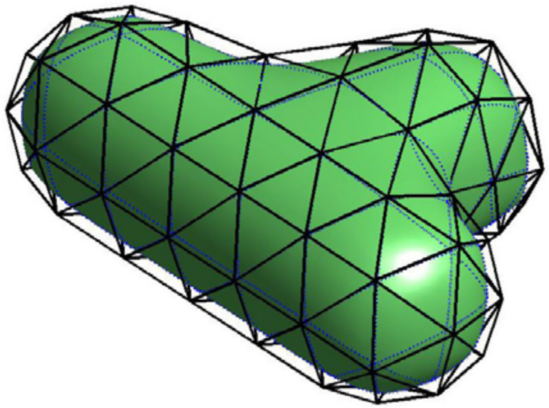
step 1



limit surface

Linear approximations

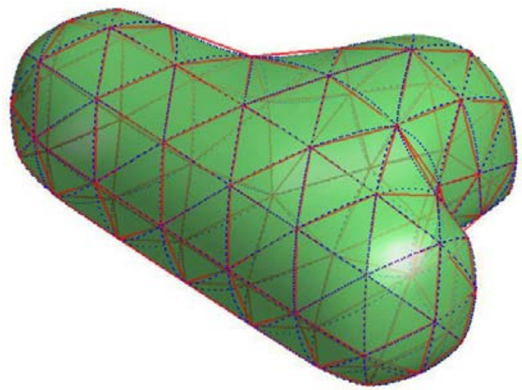
Linear approximations of a Loop surface are used in surface rendering, numerical control tool-path generation, etc



control mesh

Two linear approximations:

- (refined) control mesh
- limit mesh: obtained by pushing the control points of a control mesh to their limit positions
  - It inscribes the limit surface



limit mesh

Problems of limit mesh approx.

- Approximation error
  - How to estimate the error (distance) between a Loop surface and its limit mesh?
- Subdivision depth estimation
  - How many steps of subdivision would be necessary to satisfy a user-specified error tolerance?

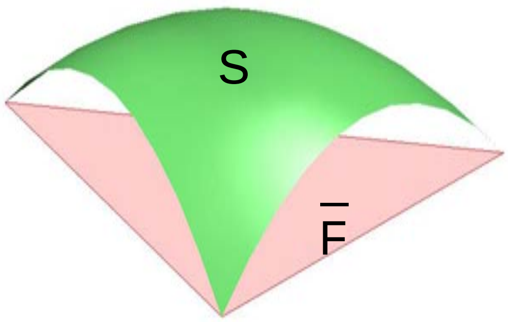
Distance (error)

Each surface patch $\mathbf{S}$ corresponds to a limit triangle $\overline{\mathbf{F}}$ in its limit mesh

- $\overline{\mathbf{F}}$ is formed by connecting the three corners of $\mathbf{S}$

Distance (error) between a patch $\mathbf{S}$ and the limit mesh is defined as

$$\max_{(v,w)\in\Omega} \left\| \mathbf{S}(v,w) - \overline{\mathbf{F}}(v,w) \right\|$$



- $\mathbf{S}(v,w)$: Stam's parameterization of $\mathbf{S}$
- $\overline{\mathbf{F}}(v,w)$: linear parameterization of $\overline{\mathbf{F}}$
- Ω : the unit triangle

$$\Omega = \left\{ (v,w) \middle| v \in [0,1], \omega \in [0,1-v] \right\}$$

Control mesh of a Loop patch

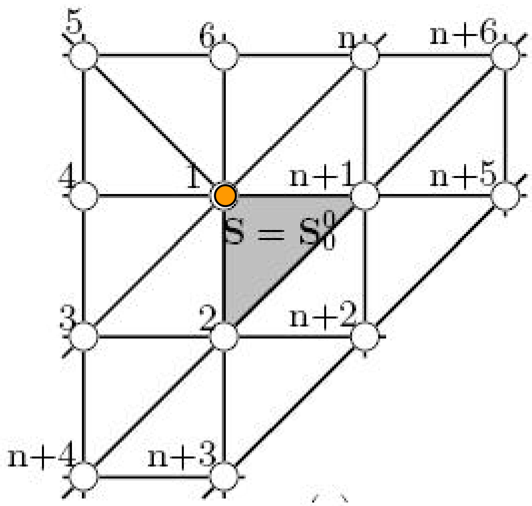
Assume: the initial control mesh has been subdivided at least once

- Each triangle face contains at most one extraordinary point

The control mesh of a Loop patch $\mathbf{S}$ of valence n consists of n+6 control points

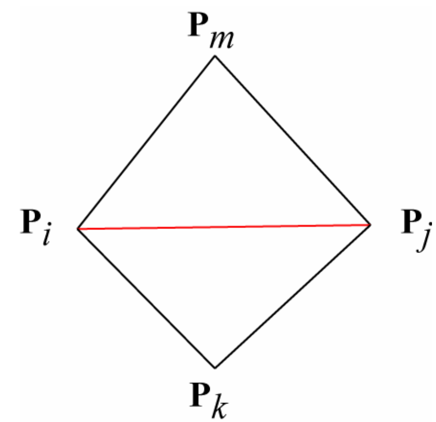
$$\{\mathbf{P}_i : i = 1, 2, \ldots, n+6\}$$

- P_1: the only extraordinary point of valence n



Control mesh of a Loop patch of valence n

Second order norm



Each interior edge $\overline{\mathbf{P}_i\mathbf{P}_j}$ in the control mesh can be associated with a mixed second difference (MSD):

$$\mathbf{P}_i + \mathbf{P}_j - \mathbf{P}_k - \mathbf{P}_m$$

The second order norm M of the control mesh of Loop patch $\mathbf{S}$ is defined as the maximum norm of n+9 MSDs:

$$\begin{aligned} M = \max\{&\|\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{n+1} - \mathbf{P}_3\|, \{\|\mathbf{P}_1 + \mathbf{P}_i - \mathbf{P}_{i-1} - \mathbf{P}_{i+1}\| : 3 \le i \le n\}, \\ &\|\mathbf{P}_1 + \mathbf{P}_{n+1} - \mathbf{P}_n - \mathbf{P}_2\|, \|\mathbf{P}_2 + \mathbf{P}_{n+1} - \mathbf{P}_1 - \mathbf{P}_{n+2}\|, \\ &\|\mathbf{P}_2 + \mathbf{P}_{n+2} - \mathbf{P}_{n+1} - \mathbf{P}_{n+3}\|, \|\mathbf{P}_2 + \mathbf{P}_{n+3} - \mathbf{P}_{n+2} - \mathbf{P}_{n+4}\|, \\ &\|\mathbf{P}_2 + \mathbf{P}_{n+4} - \mathbf{P}_{n+3} - \mathbf{P}_3\|, \|\mathbf{P}_2 + \mathbf{P}_3 - \mathbf{P}_{n+4} - \mathbf{P}_1\|, \\ &\|\mathbf{P}_{n+1} + \mathbf{P}_n - \mathbf{P}_1 - \mathbf{P}_{n+6}\|, \|\mathbf{P}_{n+1} + \mathbf{P}_{n+6} - \mathbf{P}_n - \mathbf{P}_{n+5}\|, \\ &\|\mathbf{P}_{n+1} + \mathbf{P}_{n+5} - \mathbf{P}_{n+6} - \mathbf{P}_{n+2}\|, \|\mathbf{P}_{n+1} + \mathbf{P}_{n+2} - \mathbf{P}_{n+5} - \mathbf{P}_2\|\} \\ = \max\{&\|\alpha_i\| : i = 1, \ldots, n+9\} \ . \end{aligned}$$

Our goal

To derive a bound on the distance between a Loop patch $\mathbf{S}(v,w)$ and its limit triangle $\overline{\mathbf{F}}(v,w)$ as

$$\max_{(v,w)\in\Omega}\left\|\mathbf{S}(v,w)-\overline{\mathbf{F}}(v,w)\right\|\leq\beta(n)M$$

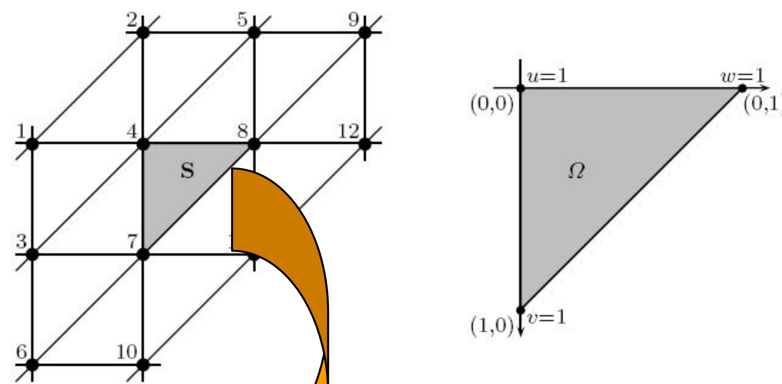
- M : the second order norm of the control mesh of $\mathbf{S}$
- $\beta(n)$: a constant only depends on valence n

Regular patch

$\mathbf{S}(v,w)$ is a quartic box spline patch defined by 12 control points:

$$\mathbf{S}(v,w) = \sum_{i=1}^{12} \mathbf{p}_i N_i(v,w), (v,w) \in \Omega$$

- $N_i(v,w)$: quartic box spline basis functions

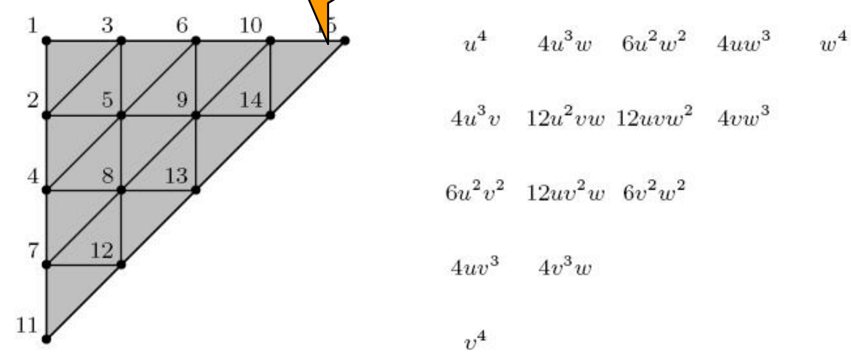


Control points of a quartic box spline patch

$\mathbf{S}(v,w)$ is also a quartic tri-angular Bézier patch:

$$\mathbf{S}(v,w) = \sum_{i=1}^{15} \mathbf{b}_i B_i(v,w), (v,w) \in \Omega$$

- $B_i(v,w)$: quartic Bernstein polynomials
- $\mathbf{b}_i$: Bézier points of $\mathbf{S}$



Bézier points of a quartic triangular Bézier patch and Bernstein polynomials

Regular patch

Limit triangle is $\overline{\mathbf{F}} = \{\mathbf{b}_1, \mathbf{b}_{11}, \mathbf{b}_{15}\}$, and its linear parameterization is

$$\overline{\mathbf{F}}(v,w) = u\mathbf{b}_1 + v\,\mathbf{b}_{11} + w\mathbf{b}_{15}, (v,w) \in \Omega$$

- (u,v,w) : barycentric system of coordinates for the unit triangle, $u = 1 - v - w$

By the linear precision property of the Bernstein polynomials, $\overline{\mathbf{F}}(v,w)$ can be expressed into the quartic Bézier form:

$$\overline{\mathbf{F}}(v,w) = \sum_{i=1}^{15} \overline{\mathbf{b}}_i B_i(v,w), (v,w) \in \Omega$$

- Here,

$$\overline{\mathbf{b}}_{\frac{(j+k)(j+k+1)}{2}+k+1} = \overline{\mathbf{F}}\left(\tfrac{j}{4}, \tfrac{k}{4}\right)$$

Regular patch

For $(u,v)\in\Omega$,

$$\left\|\mathbf{S}(v,w)-\overline{\mathbf{F}}(v,w)\right\| = \left\|\sum_{i=1}^{15}(\mathbf{b}_i-\overline{\mathbf{b}}_i)B_i(v,w)\right\|$$

$$\leq \sum_{i=1}^{15}\left\|\mathbf{b}_i-\overline{\mathbf{b}}_i\right\|B_i(v,w)$$

By solving some constrained minimization problems, $\left\|\mathbf{b}_i-\overline{\mathbf{b}}_i\right\|$ can be bounded as

$$\left\|\mathbf{b}_i-\overline{\mathbf{b}}_i\right\| \leq \delta_i M, 1\leq i\leq 15$$

- Here, M is the second order norm of $\mathbf{S}$, and

$$\delta_1=\delta_{11}=\delta_{15}=0 \qquad \delta_2=\delta_3=\delta_7=\delta_{10}=\delta_{12}=\delta_{14}=\tfrac{1}{4}$$

$$\delta_4=\delta_6=\delta_{13}=\tfrac{1}{3} \qquad \delta_5=\delta_8=\delta_9=\tfrac{5}{12}$$

Regular patch

Thus,

$$\left\|\mathbf{S}(v,w)-\overline{\mathbf{F}}(v,w)\right\| \le \sum_{i=1}^{15}\delta_i B_i(v,w)M = D(v,w)M$$

Here, $D(v,w)=\sum_{i=1}^{15}\delta_i B_i(v,w)=v+w-v^2-vw-w^2$ is called the distance bound function of $\mathbf{S}(v,w)$ with respect to $\overline{\mathbf{F}}(v,w)$.

Since $\max_{(v,w)\in\Omega} D(v,w)=\frac{1}{3}=D(\frac{1}{3},\frac{1}{3})$, we have

$$\left\|\mathbf{S}(v,w)-\overline{\mathbf{F}}(v,w)\right\| \le \tfrac{1}{3}M$$

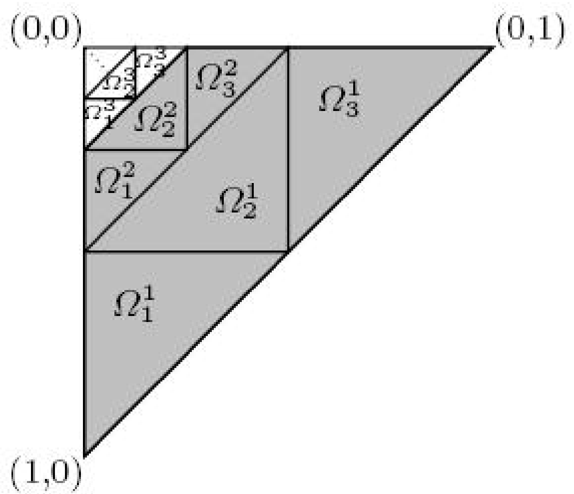
Extraordinary patch

$\mathbf{S}$ can be partitioned into an infinite sequence of regular triangular sub-patchs:

$$\{\mathbf{S}_m^k : k \ge 1, m = 1, 2, 3\}$$

corresponding to tiles $\Omega_m^k, k \ge 1, m = 1, 2, 3$

$\overline{\mathbf{F}}$ is partitioned into sub-triangles $\hat{\mathbf{F}}_m^k, k \ge 1, m = 1, 2, 3$ accordingly.



For each subpatch $\mathbf{S}_m^k$,

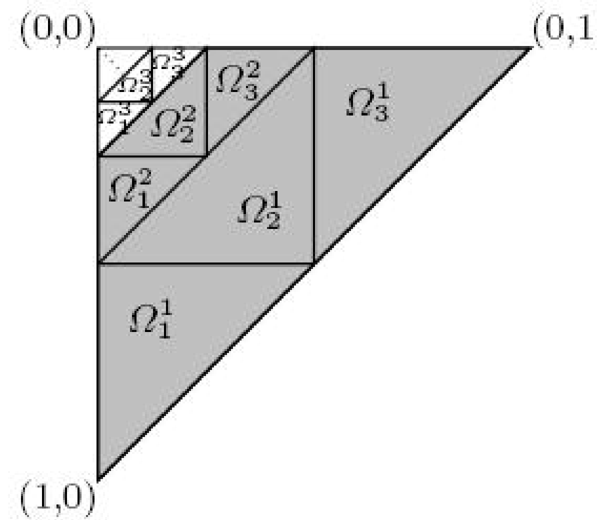
$$\left\| \mathbf{S}_m^k(v,w) - \hat{\mathbf{F}}_m^k(v,w) \right\| \le D_m^k(v,w) M \le \beta_m^k(n) M$$

- $D_m^k(v,w)$: distance bound function
- $\beta_m^k(n) = \max_{(v,w)\in\Omega} D_m^k(v,w)$

Extraordinary patch

Let $\beta(n) = \max_{k\geq 1,\, m=1,2,3} \beta_m^k(n)$, we have

$$\max_{(v,w)\in\Omega} \left\| \mathbf{S}(v,w) - \overline{\mathbf{F}}(v,w) \right\| \leq \beta(n) M$$



For $3 \leq n \leq 50$, the following fact is found:

- Maximum of $\beta_m^k(n), k \geq 1, m = 1,2,3$ appears in the domain

$$\overline{\Omega} = \Omega \cap \{(v,w) \,|\, v + w \geq \tfrac{1}{4}\}$$

To compute the values of $\beta(n)$, only four subpatchs $\mathbf{S}_1^1, \mathbf{S}_2^1, \mathbf{S}_1^2, \mathbf{S}_2^2$ are needed to be analyzed.

Subdivision depth estimation

Theorem Given an error tolerance $\varepsilon > 0$, after

$$k = \min_{0 \le j \le \lambda - 1} \lambda l_j + j$$

steps of subdivision on the control mesh of a patch S, the distance between S and its level-k control mesh is smaller than ε. Here

$$l_j = \left\lceil \log_{1/r_\lambda(n)} \frac{r_j(n) \max\{\beta(n), \frac{1}{3}\} M}{\varepsilon} \right\rceil, \; 0 \le j \le \lambda - 1, \; \lambda \ge 1$$

and $r_j(n)$ is the j-step convergence rate of second order norm.

For regular Loop patches, $k = \left\lceil \log_4 \frac{M}{3\varepsilon} \right\rceil$

Comparison

The distance between a Loop patch S and its control mesh can be bounded as:

$$\max_{(u,v)\in\Omega}\left\|\mathbf{S}(v,w)-\mathbf{F}(v,w)\right\| \leq C_\lambda(n)M\,,\ \lambda\geq 1$$

Comparison results of $C_3(n)$ and $\beta(n)$, $3 \leqslant n \leqslant 10$

n	3	4	5	6	7	8	9	10
$C_3(n)$	0.872850	0.780397	0.858623	0.500000	0.875407	0.866176	0.856245	0.847476
$\beta(n)$	0.358813	0.350722	0.342499	0.333333	0.329252	0.332001	0.333880	0.335299

A limit mesh approximates a Loop surface better than the corresponding control mesh in general.

Comparison

Set $M = 1$, error tolerance $\varepsilon = 0.0001$. Comparison results of subdivision depths for control mesh and limit mesh approximations are as follows:

n	3	4	5	6	7	8	9	10
control mesh	8	10	14	7	19	21	22	23
limit mesh	7	9	12	6	16	17	18	18

Limit mesh approximation has a 20% improvement over control mesh approximation in most of the cases.

Conclusion and remark

- We propose an approach to estimate the bound on the distance between a Loop surface and its limit mesh.
- A subdivision depth estimation formula for limit mesh approximation is derived.
- The upper bounds derived here are optimal for regular patches, but not optimal for extraordinary patches.

Thank you!