Journal of Computational Mathematics, Vol.24, No.2, 2006, ??–??.

NEW PROOF OF DIMENSION FORMULA OF SPLINE SPACES OVER T-MESHES VIA SMOOTHING COFACTORS *

Zhang-jin Huang Jian-song Deng Yu-yu Feng Fa-lai Chen
(*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China*)

Abstract

A T-mesh is basically a rectangular grid that allows T-junctions. Recently, Deng *etal* introduced splines over T-meshes, which are generalizations of T-splines invented by Sederberg *etal*, and proposed a dimension formula based on the B-net method. In this paper, we derive an equivalent dimension formula in a different form with the smoothing cofactor method.

Mathematics subject classification:
Key words: Spline space, T-mesh, Smoothing cofactors.

1. Introduction

T-meshes are formed by a set of horizontal line segments and a set of vertical line segments, where T-junctions are allowed. See Figure 1 for examples.

Traditional tensor-product B-spline functions, which are a basic tool in the design of free-form surfaces, are defined over special T-meshes, where no T-junctions appear. B-spline surfaces have the drawback that arises from the mathematical properties of the tensor-product B-spline basis functions. Two global knot vectors which are shared by all basis functions, do not allow local modification of the domain partition. Thus, if we want to construct a surface which is flat in the most part of the domain, but sharp in a small region, we have to use more control points not only in the sharp region, but also in the regions propagating from the sharp region along horizontal and vertical directions to maintain the tensor-product mesh structure. The superfluous control points are a big burden to modelling systems. In [5], Sederberg *etal* explained the troubles made by these superfluous control points in details.

To overcome this limitation, we need the local refinement of B-spline surfaces, i.e. to insert a single control point without propagating an entire row or column of control points. In [4] hierarchical B-splines were introduced, and two concepts were defined: local refinement using an efficient representation and multi-resolution editing. In principle, Hierarchical B-splines are the accumulation of tensor-product surfaces with different resolutions and domains. Weller and Hagen [8] discussed tensor-product splines with knot segments. In fact, they defined a spline space over a more general T-mesh, where crossing, T-junctional, and L-junctional vertices are allowed. But its dimensions are estimated and its basis functions are given over the mesh induced by some semi-regular basis functions.

In 2003, Sederberg *etal* [5] invented T-spline. It is a point-based spline, i.e., for every vertex, a blending function of the spline space is defined. Each of the blending functions comes from some tensor-product spline space. Though this type of splines supports many valuable operations within a consistent framework, but some of them, say, local refinement, are

* Received May 18, 2005; Final revised November 20, 2005.
1) The authors are supported by the Outstanding Youth Grant of NSF of China (No.60225002), a National Key Basic Research Project of China (2004CB318000), NSF of China (No. 60473132), the TRAPOYT in Higher Education Institute of MOE of China, and SRF for ROCS, SEM.

not simple. In the T-spline theory, the local refinement is dependent on the structure of the mesh, and its complexity is uncertain. Whether T-spline blending functions are always linearly independent is an open question [6]. The reason leading to these problems is that the spline over every cell of the mesh is not a polynomial, but a piecewise polynomial.

In [2], Deng *etal* formulated the concept of T-meshes, and studied the spline space over T-meshes. They forced the spline on every cell to be a tensor-product polynomial and achieve the specified smoothness across common edges, and derived a dimension formula when the smoothness is less than half of the degree of polynomials with a method based on B-nets.

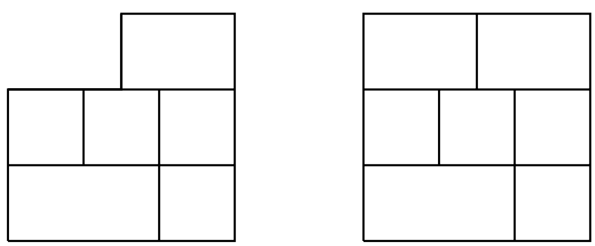
In the theory of multivariate spline, smoothing cofactor method [7] is another dominant approach to calculate the dimension of some specified spline space. In this paper, we derive a dimension formula equivalent to Deng's formula with the smoothing cofactor method. The proof is longer than the B-net version, but it is revelatory. Based on some results in this paper, we have implemented a quasi-real-time algorithm, which will be explored in another forthcoming paper, to calculate the dimension of a general spline space over T-meshes. And we expect that we can generalize Deng's formula based on the smoothing cofactor method in the future.

The paper is organized as follows. Section 2 presents a brief review of the spline spaces over T-meshes. In Section 3, by introducing the concepts of vertex cofactor and in-line, we derive a dimension formula for the spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ when $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$ with the smoothing cofactor method, and prove that it is equivalent to Deng's formula. In the final section, we conclude the paper with some further research problems.

2. Spline spaces over T-meshes

In this section, we first present some concepts related with T-meshes, and then review spline function spaces over T-meshes.

2.1 T-mesh



a b

Figure 1: Examples of T-mesh

A **T-mesh** is basically a rectangular grid that allows T-junctions [5]. The longest possible horizontal or vertical line segments to make up a T-mesh are called **grid lines**. We assume that the endpoints of each grid line in the T-mesh must be on two other grid lines, and each **cell** or **facet** (the area without any line segment inside it) in the grid must be a rectangle. Figure 1 illustrates two examples of T-meshes, while in Figure 2 two examples of non-T-meshes are shown.

A grid point in a T-mesh is also called a **vertex** of the T-mesh. If a vertex is on the boundary grid line of a T-mesh, then is called a **boundary vertex**. Otherwise, it is called an **interior vertex**. For example, b_i, $i = 1, \ldots, 10$ in Figure 3 are boundary vertices, and all the

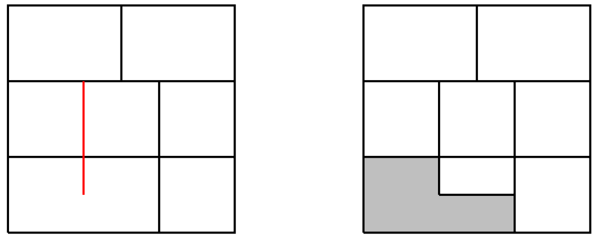


Figure 2: Examples of non-T-mesh

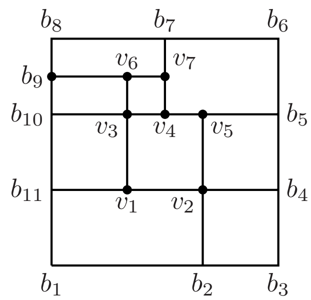


Figure 3: A T-mesh with notations

other vertices v_i, $i = 1, \ldots, 5$ are **interior vertices**. Interior vertices have two types. One is **crossing**, for example, v_2 in Figure 3; and the other is **T-junctional**, for example, v_1 in Figure 3. We call them **crossing vertices** and **T-vertices** respectively.

The line segment connecting two adjacent vertices on a grid line is called an **edge** of the T-mesh. If an edge is on the boundary of the T-mesh, then it is called a **boundary edge**; otherwise it is called an **interior edge**. For example, in Figure 3, $b_{11}v_1$ and v_1v_2 are interior edges while b_1b_2 is a boundary edge.

Except the boundary grid lines, there are three types of grid lines. We call a grid line a **cross-cut** or a **ray**, if both or only one of its endpoints lies on the boundaries, respectively. For example, in Figure 3, b_5b_{10} and b_4b_{11} are cross-cuts, while v_5b_2 , v_4b_7 and v_7b_9 are rays. Now we define the third type of grid lines. A grid line is called a **in-line**, if none of its endpoints lie on the boundaries. For example, in Figure 3, v_1v_6 is a in-line. For any grid line, it consists of one or several edges. We define its **valence** as the number of edges on the grid line.

Two cells are called **adjacent** if they share a common edge as part of their boundaries. If one cell is above(below) the other, then they are called **adjacent vertically**. If one cell is on the left(right) of the other, then they are called **adjacent horizontally**. A cell is called **adjacent** to a grid line (an edge or composition of several edges) if some boundary line of the cell is part of the grid line.

As in [2], we consider only T-meshes whose boundary grid lines form a rectangle, see Figure 1(b). We call this type of T-meshes **regular T-meshes**.

2.2 The spline space

Given a T-mesh $\mathcal{T}$, we use $\mathcal{F}$ to denote all the cells in $\mathcal{T}$ and Ω to denote the region occupied by all the cells in $\mathcal{T}$. Let

$$\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) := \Big\{ s(x,y) \in C^{\alpha,\beta}(\Omega) \Big| s(x,y)|_\phi \in \mathbb{P}_{mn}, \text{for any } \phi \in \mathcal{F} \Big\}$$

where $\mathcal{P}_{mn}$ is the space of all polynomials with bi-degree (m,n), and $C^{\alpha,\beta}(\Omega)$ the space consisting of all bivariate functions which are continuous in Ω with order α along x direction and with order β along y direction. It is obvious that $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ is a linear space. We call it **the spline space over the given T-mesh $\mathcal{T}$**.

In [2], Deng *etal* derived the following dimension formula for the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ with B-net method.

Theorem 2.1. *Given a regular T-mesh and a corresponding spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, then*

$$\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) = F(m+1)(n+1) - E_h(m+1)(\beta+1) - E_v(\alpha+1)(n+1) + V(\alpha+1)(\beta+1), \quad (1)$$

where F is the number of cells in $\mathcal{T}$, E_h and E_v the number of interior horizontal edges and the number of interior vertical edges respectively, and V the number of interior vertices.

3. The Dimension Formula

In the theory of multivariate splines, in order to calculate the dimension of some specified spline space, we first need to transfer the smoothness conditions into algebraic forms. There are many approaches to address this problem. Besides B-net method [3], smoothing cofactor method [7] is another dominant one. In this paper, we will apply this method to calculate the dimensions of spline spaces over T-meshes.

3.1 Smoothing cofactors and vertex cofactors

Suppose two adjacent facets ϕ_1 and $\phi_2 \in \mathcal{F}$, their boundary segments share a common edge e in $\mathcal{T}$, as shown in Figure 4. The common segment is vertical or horizontal, and hence, has constant x coordinate or y coordinate, respectively.

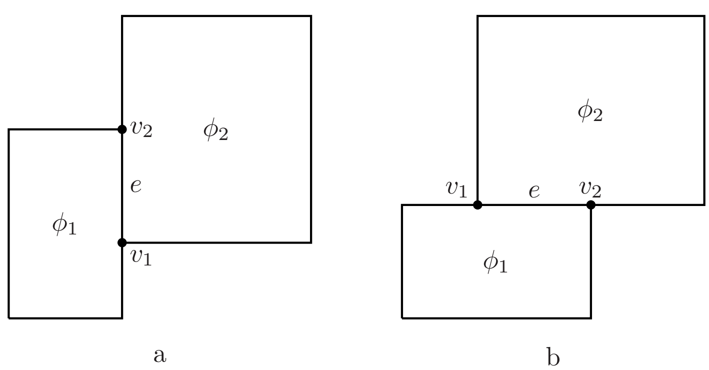


Figure 4: Two adjacent cells

Given a spline $s(x,y) \in \mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, we assume

$$s|_{\phi_1} = s_1(x,y), \qquad s|_{\phi_2} = s_2(x,y).$$

According to the smoothing cofactor theory, if the common edge e is on the straight line $x - x_0 = 0$(see Figure 4(a)), then there exists $\lambda(x,y) \in \mathbb{P}_{m-\alpha-1,n}$ such that

$$s_2(x,y) - s_1(x,y) = \lambda(x,y)(x - x_0)^{\alpha+1}. \tag{2}$$

If the common edge e is on the straight line $y - y_0 = 0$(see Figure 4(b)), then there exists $\mu(x,y) \in \mathbb{P}_{m,n-\beta-1}$ such that

$$s_2(x,y) - s_1(x,y) = \mu(x,y)(y - y_0)^{\beta+1}. \tag{3}$$

Here $\lambda(x,y)$ and $\mu(x,y)$ are called the **smoothing cofactors** of $s(x,y)$ across the corresponding edges, respectively.

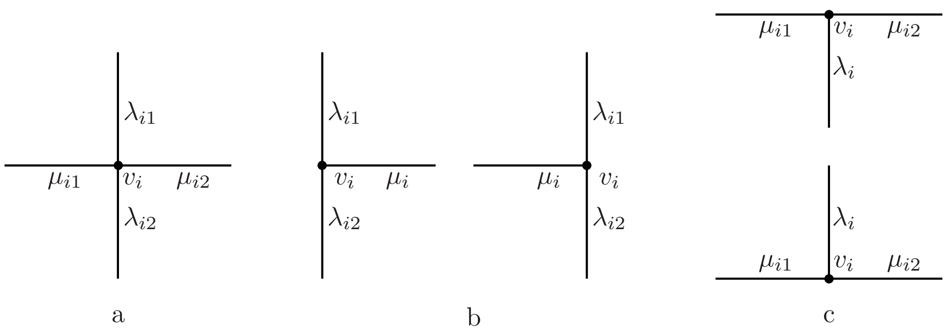


Figure 5: Conformality conditions

We use $\lambda(x,y)$ to denote the smoothing cofactor across a vertical interior edge from left to right, and $\mu(x,y)$ to denote the smoothing cofactor across a horizontal interior edge from bottom to top.

Let

$$\mathcal{V} = \{\nu_1, \dots, \nu_V\}, \quad \mathcal{E}^h = \{\varepsilon_1^h, \dots, \varepsilon_{E_h}^h\}, \quad \mathcal{E}^v = \{\varepsilon_1^v, \dots, \varepsilon_{E_v}^v\}.$$

be the set of interior vertices, interior horizontal edges and interior vertical edges, respectively. Hence the set of interior edges $\mathcal{E} = \mathcal{E}^h \cup \mathcal{E}^v$.

Suppose $\nu_i = (x_i, y_i)$ is a interior vertex of $\mathcal{T}$, which is a crossing vertex or a T-vertex(see Figure 5). The horizontal T-vertex on the left of Figure 5(b) is called a **right T-vertex**, while the one on the right of Figure 5(b) is called a **left T-vertex**. The vertical T-vertex on the top of Figure 5(c) is called a **down T-vertex**, while the one on the bottom of Figure 5(c) is called an **up T-vertex**.

If ν_i is a crossing vertex(Figure 5(a)), then the **conformality condition** of $s(x,y)$ at ν_i is

$$(\lambda_{i1}(x,y) - \lambda_{i2}(x,y))(x - x_i)^{\alpha+1} + (\mu_{i1}(x,y) - \mu_{i2}(x,y))(y - y_i)^{\beta+1} \equiv 0; \tag{4}$$

if ν_i is a horizontal T-vertex (Figure 5(b)), then the conformality condition of $s(x,y)$ at ν_i is

$$(\lambda_{i1}(x,y) - \lambda_{i2}(x,y))(x - x_i)^{\alpha+1} \mp \mu_i(x,y)(y - y_i)^{\beta+1} \equiv 0, \tag{5}$$

for a right T-vertex, the sign before $\mu_i(x,y)$ is '$-$', otherwise '+'; if ν_i is a vertical T-vertex (Figure 5(c)), then the conformality condition of $s(x,y)$ at ν_i is

$$\mp\lambda_i(x,y)(x - x_i)^{\alpha+1} + (\mu_{i1}(x,y) - \mu_{i2}(x,y))(y - y_i)^{\beta+1} \equiv 0, \tag{6}$$

for a down T-vertex, the sign before $\lambda_i(x,y)$ is '$-$', otherwise '$+$'. Here $\lambda_i(x,y)$, $\lambda_{i1}(x,y)$, $\lambda_{i2}(x,y) \in \mathbb{P}_{m-\alpha-1,n}$, and $\mu_i(x,y)$, $\mu_{i1}(x,y)$, $\mu_{i2}(x,y) \in \mathbb{P}_{m,n-\beta-1}$.

We analysis further the conformality conditions in Equation (4), (5) or (6). For Equation (4), since $(x-x_i)^{\alpha+1}$ and $(y-y_i)^{\beta+1}$ are prime, we have

$$\begin{aligned} \lambda_{i1}(x,y) - \lambda_{i2}(x,y) &= v_i(x,y)(y-y_i)^{\beta+1}, \\ \mu_{i1}(x,y) - \mu_{i2}(x,y) &= h_i(x,y)(x-x_i)^{\alpha+1} \end{aligned}$$

for some $h_i(x,y), v_i(x,y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$. Putting these back into Equation (4) gives

$$(v_i(x,y) + h_i(x,y))(x-x_i)^{\alpha+1}(y-y_i)^{\beta+1} \equiv 0.$$

Hence, $v_i(x,y) = -h_i(x,y) = c_i(x,y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$. We call $c_i(x,y)$ the **vertex cofactor** of $s(x,y)$ corresponding to an interior vertex ν_i.

Then the conformality condition at a crossing vertex in Equation (4) can be rewritten as

$$\begin{aligned} \lambda_{i1}(x,y) - \lambda_{i2}(x,y) &= c_i(x,y)(y-y_i)^{\beta+1}, \\ \mu_{i1}(x,y) - \mu_{i2}(x,y) &= -c_i(x,y)(x-x_i)^{\alpha+1}. \end{aligned} \tag{7}$$

Similarly, the conformality condition at a horizontal T-vertex in Equation (5) can be rewritten as

$$\begin{aligned} \lambda_{i1}(x,y) - \lambda_{i2}(x,y) &= c_i(x,y)(y-y_i)^{\beta+1}, \\ \mp\mu_i(x,y) &= -c_i(x,y)(x-x_i)^{\alpha+1}; \end{aligned} \tag{8}$$

and the conformality condition at a vertical T-vertex in Equation (6) can be rewritten as

$$\begin{aligned} \mp\lambda_i(x,y) &= c_i(x,y)(y-y_i)^{\beta+1}, \\ \mu_{i1}(x,y) - \mu_{i2}(x,y) &= -c_i(x,y)(x-x_i)^{\alpha+1}. \end{aligned} \tag{9}$$

We note that, even if all the vertex cofactors $c_i(x,y), i = 1,\ldots,V$ are determined, for each cross-cut, there is still one interior edge whose smoothing cofactor is completely free.

Let E_h^c and E_v^c be the number of horizontal cross-cuts and the number of vertical cross-cuts respectively, and σ be the number of independent free coefficients of the vertex cofactors. Then the number of free coefficients of the smoothing cofactors of all interior edges is

$$E_h^c(m+1)(n-\beta) + E_v^c(m-\alpha)(n+1) + \sigma,$$

hence, we have the following dimension formula of the spline space over a T-mesh:

$$\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) = (m+1)(n+1) + E_h^c(m+1)(n-\beta) + E_v^c(m-\alpha)(n+1) + \sigma \tag{10}$$

3.2 In-line conformality conditions

If there are no in-lines in $\mathcal{T}$, namely, $\mathcal{T}$ is a simple quasi-cross-cut partition, the vertex cofactors are relatively independent. Then we have

$$\sigma = V(m-\alpha)(n-\beta),$$

and the dimension formula (10) becomes

$$\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) = (m+1)(n+1) + E_h^c(m+1)(n-\beta) + E_v^c(m-\alpha)(n+1) + V(m-\alpha)(n-\beta)$$

If there exist in-lines in $\mathcal{T}$, the vertex cofactors may need to satisfy some constraints. So in general,

$$\sigma \leqslant V(m-\alpha)(n-\beta).$$

For a horizontal in-line, both the endpoints are horizontal T-vertices. Suppose its valence is $\kappa(\kappa \geqslant 1)$, the smoothing cofactors across κ interior edges ε_{k_t} are $\mu_{k_t}(x,y), t = 1, \ldots, \kappa$, and the vertex cofactors corresponding to $\kappa+1$ interior vertices $\nu_{i_t} = (x_{i_t}, y_i)$ are $c_{i_t}(x,y), t = 0, \ldots, \kappa$(see Figure 6). By Equation (7) and (8), we have the following $\kappa+1$ equations

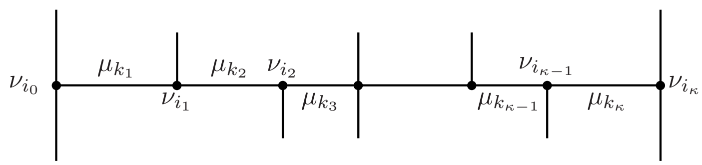


Figure 6: Horizontal in-line

$$\begin{aligned}
-\mu_{k_1}(x,y) &= c_{i_0}(x,y)(x-x_{i_0})^{\alpha+1} \\
\mu_{k_1}(x,y) - \mu_{k_2}(x,y) &= c_{i_1}(x,y)(x-x_{i_1})^{\alpha+1} \\
&\cdots\cdots \\
\mu_{k_{\kappa-1}}(x,y) - \mu_{k_\kappa}(x,y) &= c_{i_{\kappa-1}}(x,y)(x-x_{i_{\kappa-1}})^{\alpha+1} \\
\mu_{k_\kappa}(x,y) &= c_{i_\kappa}(x,y)(x-x_{i_\kappa})^{\alpha+1}
\end{aligned}$$

Sum the above $\kappa+1$ equations together, we obtain the **horizontal in-line conformality condition** of the corresponding in-line

$$\sum_{t=0}^{\kappa} c_{i_t}(x,y)(x-x_{i_t})^{\alpha+1} \equiv 0 \tag{11}$$

For a vertical in-line of valence $\iota(\iota \geqslant 1)$, suppose $\iota+1$ interior vertices' y-coordinates are $y_{j_t}, t = 0, \ldots, \iota$, and corresponding vertex cofactors are $c_{j_t}(x,y), t = 0, \ldots, \iota$. Similarly, we have the following **vertical in-line conformality condition**

$$\sum_{t=0}^{\iota} c_{j_t}(x,y)(y-y_{j_t})^{\beta+1} \equiv 0 \tag{12}$$

Putting together all the in-line conditions like Equation (11) or (12) of the in-lines of $\mathcal{T}$, we get the **global in-line conformality condition** of $s(x,y)$. The global in-line conformality condition is a system of homogeneous linear equations with the coefficients of the vertex cofactors as unknows.

Let E_h^i, E_v^i be the number of horizontal in-lines and the number of vertical in-lines in $\mathcal{T}$, respectively. Then the global in-line conformality condition can be written as the following matrix form:

$$\mathbf{M}Z = 0, \tag{13}$$

where Z is a column vector consisting of the $V(m-\alpha)(n-\beta)$ coefficients of the vertex cofactors corresponding to the V interior vertices, and $\mathbf{M}$ is a $(E_h^i(m+1)(n-\beta)+E_v^i(m-\alpha)(n+1)) \times V(m-\alpha)(n-\beta)$ matrix. Then σ can be determined as

$$\sigma = V(m-\alpha)(n-\beta) - \operatorname{rank}\mathbf{M}. \tag{14}$$

3.3 Dimension formula

Suppose $p > q \geqslant 1$, define $\mathcal{A}_{p,q}(x) = (a_{i,j})$ as a $(p+1)\times(p-q)$ matrix, where

$$a_{i,j} = \begin{cases} 0, & i < j \text{ or } i > j+q+1; \\ \binom{q+1}{l} x^{q+1-l}, & j \leqslant i \leqslant j+q+1,\ l = i-j. \end{cases}$$

Nonzero elements in each column of $\mathcal{A}_{p,q}$ are the expanding terms of binomial $(x+1)^{q+1}$, that is,

$$\mathcal{A}_{p,q}(x) = \begin{pmatrix} x^{q+1} & & & \\ \binom{q+1}{1}x^q & x^{q+1} & & \\ \vdots & \vdots & \ddots & \\ 1 & \binom{q+1}{q}x & \cdots & x^{q+1} \\ & 1 & \cdots & \binom{q+1}{1}x^q \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix},$$

We have the following

Lemma 3.1. *Let* $\mathbf{A}_{p,q}(x,y) = \begin{pmatrix} \mathcal{A}_{p,q}(x) & \mathcal{A}_{p,q}(y) \end{pmatrix}$, *suppose* $p \geqslant 2q+1$ *and* $x \neq y \in \mathbb{R}$, *then*

$$\operatorname{rank}\mathbf{A}_{p,q}(x,y) = p+1.$$

Proof. Without loss of generality, assume $x \neq 0$.

When $p = 2q+1$,

$$\mathbf{A}_{p,q}(x,y) = \begin{pmatrix} x^{q+1} & & & & y^{q+1} & & & \\ \binom{q+1}{1}x^q & x^{q+1} & & & \binom{q+1}{1}y^q & y^{q+1} & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ 1 & \binom{q+1}{q}x & \cdots & x^{q+1} & 1 & \binom{q+1}{q}y & \cdots & y^{q+1} \\ & 1 & \cdots & \binom{q+1}{1}x^q & & 1 & \cdots & \binom{q+1}{1}y^q \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & 1 & & & & 1 \end{pmatrix}$$

is a $(p+1)\times(p+1)$ square matrix. In order to calculate $\det\mathbf{A}_{p,q}(x,y)$, consider $f(z) = (x+z)^{q+1}, g(z) = (y+z)^{q+1}$ two polynomials, according to the result about the resultant of two polynomials in [1, p.73], we have

$$\det\mathbf{A}_{p,q}(x,y) = \operatorname{res}(f,g,z) = (x-y)^{(q+1)^2}.$$

Since $x \neq y$, $\det\mathbf{A}_{p,q}(x,y) \neq 0$, namely, $\operatorname{rank}\mathbf{A}_{p,q}(x,y) = p+1$.

When $p > 2q+1$,

$$
\mathbf{A}_{p,q}(x,y) = \begin{pmatrix}
x^{q+1} & & & & & & y^{q+1} & & & & & \\
\vdots & \ddots & & & & & \vdots & \ddots & & & & \\
 & & x^{q+1} & & & & & & y^{q+1} & & & \\
\vdots & \ddots & \vdots & \ddots & & & \vdots & \ddots & \vdots & \ddots & & \\
1 & & & & & & 1 & & & & & \\
 & \ddots & \vdots & \ddots & \ddots & & & \ddots & \vdots & \ddots & \ddots & \\
 & & 1 & \cdots & \cdots & x^{q+1} & & & 1 & \cdots & \cdots & y^{q+1} \\
 & & & \ddots & & \vdots & & & & \ddots & & \vdots \\
 & & & & \ddots & \vdots & & & & & \ddots & \vdots \\
 & & & & & 1 & & & & & & 1
\end{pmatrix}.
$$

Delete the first $p-2q-1$ columns of matrix $\mathcal{A}_{p,q}(y)$ and denote the result matrix by $\overline{\mathcal{A}}_{p,q}(y)$, then matrix

$$
\begin{aligned}
\overline{\mathbf{A}}_{p,q} &= \begin{pmatrix} \mathcal{A}_{p,q}(x) & \overline{\mathcal{A}}_{p,q}(y) \end{pmatrix} \\
&= \begin{pmatrix}
x^{q+1} & & & & & & & & & \\
\vdots & \ddots & & & & & & & & \\
 & & x^{q+1} & & & & y^{q+1} & & & \\
\vdots & \ddots & \vdots & \ddots & & & \vdots & \ddots & & \\
1 & & & & & & & & & \\
 & \ddots & \vdots & \ddots & \ddots & & \vdots & \ddots & \ddots & \\
 & & 1 & \cdots & \cdots & x^{q+1} & 1 & \cdots & \cdots & y^{q+1} \\
 & & & \ddots & & \vdots & & \ddots & & \vdots \\
 & & & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & & & 1 & & & & 1
\end{pmatrix}
\end{aligned}
$$

is a $(p+1)\times(p+1)$ square matrix, it is easy to know the determinant

$$
\det \overline{\mathbf{A}}_{p,q} = x^{(q+1)(p-2q-1)}\mathrm{res}(f,g,z) = x^{(q+1)(p-2q-1)}(x-y)^{(q+1)^2} \neq 0,
$$

hence, $\operatorname{rank} \mathbf{A}_{p,q}(x,y) = p+1$.

It is impossible for regular T-meshes to have only in-lines, since the number of adjacent edges of an interior vertex should be 3 or 4, corresponding to T-junctional and crossing vertices respectively. Furthermore, the endpoints of an in-line can't be endpoints of other in-lines, but can be inner vertices of others. Then we have

$$
V > 2(E_h^i + E_v^i),
$$

where V is the number of interior vertices, E_h^i and E_v^i the number of horizontal in-lines and the number of vertical in-lines, respectively. So when $m \geqslant 2\alpha+1, n \geqslant 2\beta+1$, it follows that

$$
V(m-\alpha)(n-\beta) > E_h^i(m+1)(n-\beta) + E_v^i(m-\alpha)(n+1).
$$

Hence, the coefficient matrix $\mathbf{M}$ in Equation(13) has more columns than rows. And when $m \geqslant 2\alpha+1, n \geqslant 2\beta+1$, we have

Theorem 3.2. *Let* $\mathbf{M}$ *be the coefficient matrix of the global in-line conformality condition* $\mathbf{M}Z = 0$*, suppose* $m \geqslant 2\alpha+1, n \geqslant 2\beta+1$*, then*

$$\operatorname{rank}\mathbf{M} = E_h^i(m+1)(n-\beta) + E_v^i(m-\alpha)(n+1),$$

where E_h^i *and* E_v^i *are the number of horizontal in-lines and the number of vertical in-lines, respectively.*

Proof. According to the property of $\mathbf{M}$, we only need to show $\mathbf{M}$ is row full rank.

Suppose the form of the vertex cofactor $c_i(x,y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$ be

$$c_i(x,y) = \sum_{j=0}^{m-\alpha-1}\sum_{k=0}^{n-\beta-1} c_{jk}^i x^j y^k, \quad i = 1,\ldots,V.$$

For a horizontal in-line conformality condition(Equation (11)), let

$$\begin{aligned}
C_k^i &= (c_{0,k}^i, c_{1,k}^i, \ldots, c_{m-\alpha-1,k}^i)^T, \quad k = 0,1,\ldots,n-\beta-1\\
C^i &= (C_0^i, C_1^i, \ldots, C_{n-\beta-1}^i)^T,\\
C_k &= (C_k^{i_0}, C_k^{i_1}, \ldots, C_k^{i_\kappa})^T,\\
C &= (C^{i_0}, C^{i_1}, \ldots, C^{i_\kappa})^T.
\end{aligned}$$

Given $t(t=0,1,\ldots,\kappa)$ and $k(k=0,1,\ldots,n-\beta-1)$, let $\mathcal{A}_{m,\alpha}(x_{i_t})C_k^{i_t} = (b_0, b_1, \ldots, b_m)^T$, where for $j = 0,\ldots,m$,

$$\begin{aligned}
b_j &= \sum_{l=0}^{m-\alpha-1} a_{j,l} c_{l,k}^{i_t}\\
&= \sum_{l=\max(0,j-\alpha-1)}^{\min(j,m-\alpha-1)} \binom{\alpha+1}{j-l} x_{i_t}^{\alpha+1-(j-l)} c_{l,k}^{i_t},
\end{aligned}$$

since $0 \leqslant l \leqslant m-\alpha-1$, and $a_{j,l}$ is nonzero when $l \leqslant j \leqslant l+\alpha+1$.

Then we have

$$\begin{aligned}
\sum_{t=0}^{\kappa} c_{i_t}(x,y)(x-x_{i_t})^{\alpha+1} &= \sum_{t=0}^{\kappa}\sum_{j=0}^{m-\alpha-1}\sum_{k=0}^{n-\beta-1} c_{jk}^{i_t} x^j y^k (x-x_{i_t})^{\alpha+1}\\
&= \sum_{t=0}^{\kappa}\sum_{j=0}^{m-\alpha-1}\sum_{k=0}^{n-\beta-1}\sum_{l=0}^{\alpha+1}\binom{\alpha+1}{l}(-x_{i_t})^{\alpha+1-l} c_{jk}^{i_t} x^{j+l} y^k\\
&= \sum_{k=0}^{n-\beta-1} y^k \sum_{t=0}^{\kappa}\sum_{j=0}^{m-\alpha-1}\sum_{l=0}^{\alpha+1}\binom{\alpha+1}{l}(-x_{i_t})^{\alpha+1-l} c_{jk}^{i_t} x^{j+l}\\
&= \sum_{k=0}^{n-\beta-1} y^k \sum_{t=0}^{\kappa}\sum_{j=0}^{m}\sum_{l=\max(0,j-\alpha-1)}^{\min(j,m-\alpha-1)}\binom{\alpha+1}{j-l}(-x_{i_t})^{\alpha+1-(j-l)} c_{l,k}^{i_t} x^j\\
&= \sum_{k=0}^{n-\beta-1} y^k \left(\sum_{t=0}^{\kappa}\mathcal{A}_{m,\alpha}(-x_{i_t})C_k^{i_t}\right)\cdot X\\
&= \sum_{k=0}^{n-\beta-1} y^k \left(\begin{pmatrix}\mathcal{A}_{m,\alpha}(-x_{i_0}) & \mathcal{A}_{m,\alpha}(-x_{i_1}) & \ldots & \mathcal{A}_{m,\alpha}(-x_{i_\kappa})\end{pmatrix} C_k\right)\cdot X,
\end{aligned}$$

where '$\cdot$' is the dot product of vectors, $X = (1, x, \ldots, x^m)^T$.

If the sum of polynomials is expanded in ascendant lex order($x < y$), then Equation (11) can be written into

$$\mathbf{B}_{m,\alpha} C = 0,$$

where

$$\mathbf{B}_{m,\alpha} = \begin{pmatrix} \mathcal{B}_{m,\alpha}(x_{i_0}) & \mathcal{B}_{m,\alpha}(x_{i_1}) & \ldots & \mathcal{B}_{m,\alpha}(x_{i_\kappa}) \end{pmatrix}$$

and $\mathcal{B}_{m,\alpha}(x)$ is a diagonal block matrix of order $n - \beta$ as follows:

$$\mathcal{B}_{m,\alpha}(x) = \begin{pmatrix} \mathcal{A}_{m,\alpha}(-x) & & & \\ & \mathcal{A}_{m,\alpha}(-x) & & \\ & & \ddots & \\ & & & \mathcal{A}_{m,\alpha}(-x) \end{pmatrix}.$$

Similarly, for a vertical in-line conformality condition(Equation (12)), if the sum of polynomials is expanded in ascendant lex order($x > y$) and the coefficients of $c_i(x, y)$ are ordered in a different way, namely, let

$$\begin{aligned} \overline{C}^i_k &= (c^i_{k,0}, c^i_{k,1}, \ldots, c^i_{k,n-\beta-1})^T, \quad k = 0, 1, \ldots, m - \alpha - 1 \\ \overline{C}^i &= (\overline{C}^i_0, \overline{C}^i_1, \ldots, \overline{C}^i_{m-\alpha-1})^T, \\ \overline{C} &= (\overline{C}^{i_0}, \overline{C}^{i_1}, \ldots, \overline{C}^{i_\iota})^T. \end{aligned}$$

then Equation (12) can be written into

$$\mathbf{B}_{n,\beta} \overline{C} = 0,$$

where

$$\mathbf{B}_{n,\beta} = \begin{pmatrix} \mathcal{B}_{n,\beta}(y_{i_0}) & \mathcal{B}_{n,\beta}(y_{i_1}) & \ldots & \mathcal{B}_{n,\beta}(y_{i_\iota}) \end{pmatrix}$$

and $\mathcal{B}_{n,\beta}(y)$ is a diagonal block matrix of order $m - \alpha$ as follows:

$$\mathcal{B}_{n,\beta}(y) = \begin{pmatrix} \mathcal{A}_{n,\beta}(-y) & & & \\ & \mathcal{A}_{n,\beta}(-y) & & \\ & & \ddots & \\ & & & \mathcal{A}_{n,\beta}(-y) \end{pmatrix}.$$

Obviously, the ordering of in-line conformality conditions in the system of linear equations $\mathbf{M}Z = 0$, the ordering of monomials in the expanded polynomials, the ordering of vertex cofactors in Z and the ordering of coefficients of a vertex cofactor in Z influence the form of coefficient matrix $\mathbf{M}$.

We choose these orderings as follows:

1. **Ordering of in-line conformality conditions**: Since the endpoints of an in-line can't be endpoints of other in-lines, but can be inner vertices of others. Then in the global in-line conformality condition, first put the in-line conformality conditions, associated with which the in-lines whose endpoints are not in other in-lines; then in the rest, choose the in-lines whose endpoints are not in other in-lines; repeat the above process until all the in-lines are ordered.

2. **Ordering of monomials**: For a horizontal in-line conformality condition, the sum of polynomials is expanded in ascendant lex order($x < y$); for a vertical in-line conformality condition, the sum of polynomials is expanded in ascendant lex order($x > y$).

3. **Ordering of vertex cofactors**: In Z, first put the coefficients of two vertex cofactors associated with the endpoints of all in-lines sequently, then put the coefficients of other vertex cofactors arbitrarily.

4. **Ordering of coefficients of vertex cofactors**: In Z, the coefficients of vertex cofactors associated with the two endpoints are ordered in the way discussed before, the coefficients of other vertex cofactors can be ordered arbitrarily.

Then the columns corresponding to the endpoints of all the in-lines appear in the first $2(E_h^i + E_v^i)(m-\alpha)(n-\beta)$ columns of $\mathbf{M}$, and the sub-matrix consisting of these columns is with the form as the following $(E_h^i + E_v^i) \times (E_h^i + E_v^i)$ upper triangular block matrix:

$$\overline{\mathbf{M}} = \begin{pmatrix} \mathbf{B}_1(t_1, t_1') & & * \\ & \ddots & \\ & & \mathbf{B}_1(t_{E_h^i+E_v^i}, t'_{E_h^i+E_v^i}) \end{pmatrix},$$

where $\mathbf{B}_1(t,t') = \begin{pmatrix} \mathcal{B}(t) & \mathcal{B}(t') \end{pmatrix}$, $\mathcal{B}(t)$ is $\mathcal{B}_{m,\alpha}(x)$ or $\mathcal{B}_{n,\beta}(y)$, t and t' are the x-coordinates or y-coordinates of the two endpoints corresponding to the horizontal in-line and the vertical in-line respectively. In $\overline{\mathbf{M}}$, every block row corresponds to an in-line conformality condition, every block column corresponds to the two vertex cofactors associated with the endpoints of an in-line.

Rearranging columns, $\mathbf{B}_1(t,t')$ can be transformed into a diagonal block matrix:

$$\begin{pmatrix} \mathbf{A}(t,t') & & & \\ & \mathbf{A}(t,t') & & \\ & & \ddots & \\ & & & \mathbf{A}(t,t') \end{pmatrix},$$

where $\mathbf{A}(t,t')$ is $\mathbf{A}_{m,\alpha}(t,t')$ or $\mathbf{A}_{n,\beta}(t,t')$ defined in Lemma 3.1 corresponding to the horizontal in-line and the vertical in-line respectively. By Lemma 3.1, for a horizontal in-line, we have $\operatorname{rank}\mathbf{B}_1(t,t') = (m+1)(n-\beta)$; for a vertical in-line, we have $\operatorname{rank}\mathbf{B}_1(t,t') = (m-\alpha)(n+1)$. Then $\operatorname{rank}\overline{\mathbf{M}} = E_h^i(m+1)(n-\beta) + E_v^i(m-\alpha)(n+1)$, so $\mathbf{M}$ is row full rank. This completes the proof.

By the above theorem and Equation (10) and (14), we have the following dimension formula:

Theorem 3.3. *Given a regular T-mesh and a corresponding spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, then*

$$\begin{aligned} \dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) &= (m+1)(n+1) + V(m-\alpha)(n-\beta) \\ &\quad +(E_h^c - E_h^i)(m+1)(n-\beta) + (E_v^c - E_v^i)(m-\alpha)(n+1), \end{aligned} \tag{15}$$

where V is the number of interior vertices, E_h^c and E_v^c the number of horizontal cross-cuts and the number of vertical cross-cuts respectively, E_h^i and E_v^i the number of horizontal in-lines and the number of vertical in-lines respectively.

With Euler's formula, we have

Corollary 3.4. *Given a regular T-mesh and a corresponding spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, then dimension formula (1) and (15) are equivalent.*

Proof. Let υ be the number of interior vertices in a grid line(cross-cut, ray or in-line). Then for a cross-cut of valence p, we have $p = \upsilon + 1$; for a ray of valence q, we have $q = \upsilon$; for a in-line of valence r, we have $r = \upsilon - 1$.

It is easy to show that

$$E_h = \sum_{i=1}^{E_h^c} p_i + \sum_{i=1}^{E_h^r} q_i + \sum_{i=1}^{E_h^i} r_i$$

and

$$V = \sum_{i=1}^{E_h^c} (p_i - 1) + \sum_{i=1}^{E_h^r} q_i + \sum_{i=1}^{E_h^i} (r_i + 1),$$

where V is the number of interior vertices, E_h the number of interior horizontal edges, E_h^c, E_h^r and E_h^i are the number of horizontal cross-cuts, the number of horizontal rays and the number of horizontal in-lines respectively. From the above two equations, it follows that

$$E_h - V = E_h^c - E_h^i.$$

Similarly,

$$E_v - V = E_v^c - E_v^i,$$

where E_v is the number of interior vertical edges, E_v^c and E_v^i are the number of vertical cross-cuts and the number of vertical in-lines respectively.

Putting the above two equations into Equation (15), we have

$$\begin{aligned}\dim \mathcal{S}(m, n, \alpha, \beta, \mathcal{T}) &= (E_h + E_v - V + 1)(m + 1)(n + 1) + V(\alpha + 1)(\beta + 1) \\ &\quad - E_h(m + 1)(\beta + 1) - E_v(\alpha + 1)(n + 1).\end{aligned}$$

By Euler's formula, it follows that $F = E_h + E_v - V + 1$. This completes the proof.

4. Conclusions and Future Work

In this paper, we investigate the dimension of the spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ over a given T-mesh $\mathcal{T}$ with the smoothing cofactor method, and derive an explicit dimension formula when $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, which is equivalent to Deng's. However, since each grid line consists of one or several interior edges, our formula is more convenient when calculating the dimension.

Compared with the simple cross-cut partition grid in the traditional multivariate spline theory [7], besides cross-cuts and rays, there are in-lines in a T-mesh. Therefore, it is not surprising that the dimension formulae of the bivariate spline spaces on a simple cross-cut partition grid are very similar to those of the spline spaces over a T-mesh proposed here. In both spline spaces, cross-cuts contribute additional degrees of freedom, while rays do not. And in the spline spaces over a T-mesh, the existence of the in-lines decreases the degrees of freedom. We hope that the introduction of the vertex cofactor and the in-line conformality condition may give some clues to the study of the bivariate spline spaces over a T-mesh.

In our analysis procedure, the constraints $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$ are key to guarantee the coefficient matrix $\mathbf{M}$ to be row full rank. In further research, we will investigate whether a general dimension formula without the constraints can be derived.

In [2], the construction of basis functions of the spline space over a T-mesh is briefly discussed. In the future we will work on how to construct more applicable basis functions and apply them in geometric modelling, for example, in surface approximation and interpolation. The expectation is that the new splines over T-meshes inherit the attractive properties of T-spline but will prove more powerful in certain geometric operations.

References

[1] David Cox, John Little, and Donal O'Shea, Using Algebraic Geometry, GTM 85, Springer-Verlag New York, Inc., 1998.

[2] Jiansong Deng, Falai Chen, and Yuyu Feng, Dimensions of spline spaces over T-meshes, accepted by Journal of Computational and Applied Mathematics, available online on 26 August 2005.

[3] Gerald Farin, Curves and Surfaces for CAGD — A Practical Guide, 5th Edition, Morgan Kaufmann Publishers, 2002.

[4] David Forsey and Richard H. Bartels, Hierarchical B-spline refinement, *Computer Graphics*, **22**:4(1988), 205–212.

[5] Thomas W. Sederberg, Jianmin Zheng, Almaz Bakenov, and Ahmad Nasri, T-splines and T-NURCCs, *ACM Transactions on Graphics*, **22**:3(2003), 161–172.

[6] Thomas W. Sederberg, David L. Cardon, G. Thomas Finnigan, Nicholas S. North, Jianmin Zheng, and Tom Lyche, T-spline simplification and local refinement, it ACM Transactions on Graphics, **23**:3(2004), 276–283.

[7] Ren-Hong Wang, Multivariate Spline Functions and Their Applications, Kluwer Academic Publishers, 2001.

[8] Frank Weller and Hans Hagen, Tensor-product spline spaces with knot segments, Mathematical Methods for Curves and Surfaces, Ed. M. Dæhlen, T. Lyche, and L. L. Schumaker, Vanderbilt Unversity Press, 1995, 563–572.