Non-Uniform Recursive Doo-Sabin Surfaces

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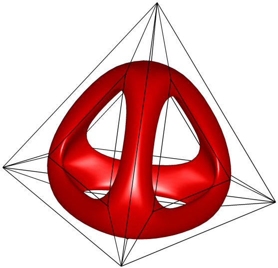
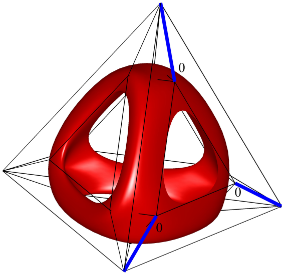
Doo-Sabin Surfaces

- Generalization of uniform biquadratic B-spline surfaces to meshes of arbitrary topology [Doo and Sabin 1978].
- Limit point rule: For an n-sided face, its centroid is the limit position of its associated extraordinary point.
  - The extraordinary points are at the "centers" of n-sided faces.
- Convergence: The Doo-Sabin refinement is convergent for extraordinary points with arbitrary valence.



Quadratic NURSSes

- Generalization of non-uniform biquadratic B-spline surfaces to meshes of arbitrary topology [Sederberg et al. 1998].
- No closed form limit point rules.
- Converge for n-sided faces with $n \leq 12$, but may diverge if $n > 12$ [Qin et al. 1998].



Doo-Sabin Subdivision

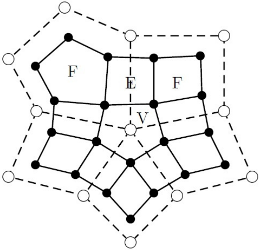
$$\overline{\mathbf{P}}_i = \sum_{j=0}^{n-1} w_{ij}\mathbf{P}_j, i = 0, \ldots, n-1.$$

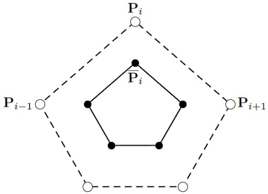
- Doo-Sabin version [Doo and Sabin 1978], extended to quadratic NURSS:

$$w_{ij} = \begin{cases} \frac{n+5}{4n}, & i = j \\ \frac{3+2\cos(2\pi(i-j)/n)}{4n}, & i \neq j \end{cases}$$

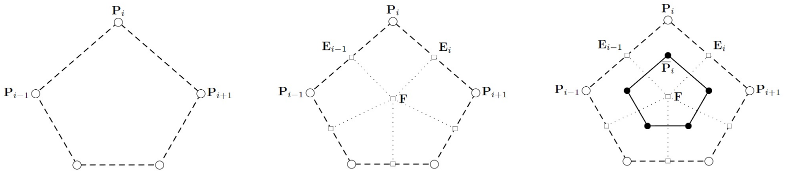
- Catmull-Clark variant [Catmull and Clark 1978]:

$$w_{ij} = \begin{cases} \frac{1}{2} + \frac{1}{4n}, & |i-j| = 0 \\ \frac{1}{8} + \frac{1}{4n}, & |i-j| = 1 \\ \frac{1}{4n}, & |i-j| > 1 \end{cases}$$





Catmull-Clark Variant of Doo-Sabin Subdivision



Repeated averaging [Stam 2001, Zorin and Schröder 2001]:

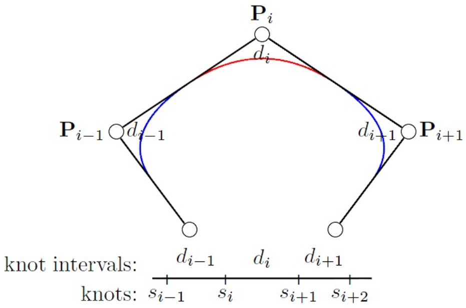
- Linear subdivision:

$$\mathbf{E}_i = \frac{1}{2}(\mathbf{P}_i + \mathbf{P}_{i+1}), \qquad \mathbf{F} = \frac{1}{n}\sum_{j=0}^{n-1}\mathbf{P}_j$$

- Dual averaging:

$$\overline{\mathbf{P}}_i = \frac{1}{4}(\mathbf{P}_i + \mathbf{E}_{i-1} + \mathbf{E}_i + \mathbf{F})$$
$$= (\frac{1}{2} + \frac{1}{4n})\mathbf{P}_i + (\frac{1}{8} + \frac{1}{4n})(\mathbf{P}_{i+1} + \mathbf{P}_{i-1}) + \frac{1}{4n}\sum \mathbf{P}_j.$$

Non-uniform Quadratic B-spline Curves

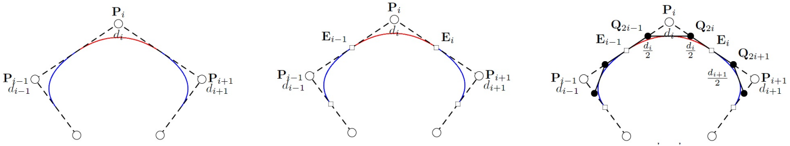


For a quadratic B-spline curve, a knot interval d_i is assigned to each control point $\mathbf{P}_i$.

- A knot interval is the difference between two adjacent knots in the knot vector, i.e., the parameter length of a B-spline curve segment.

Non-uniform Quadratic B-spline Subdivision

Refinement rules



Repeated averaging:

- Non-uniform linear subdivision:

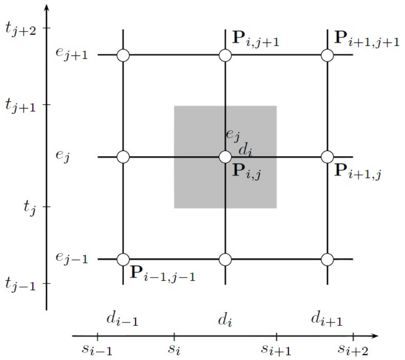
$$\mathbf{E}_i = \frac{d_{i+1}\mathbf{P}_i + d_i\mathbf{P}_{i+1}}{d_i + d_{i+1}}$$

- Averaging:

$$\mathbf{Q}_{2i} = \frac{1}{2}(\mathbf{P}_i + \mathbf{E}_i) == \frac{(d_i + 2d_{i+1})\mathbf{P}_i + d_i\mathbf{P}_{i+1}}{2(d_i + d_{i+1})}$$

$$\mathbf{Q}_{2i+1} = \frac{1}{2}(\mathbf{P}_{i+1} + \mathbf{E}_i) = \frac{d_{i+1}\mathbf{P}_i + (2d_i + d_{i+1})\mathbf{P}_{i+1}}{2(d_i + d_{i+1})}.$$

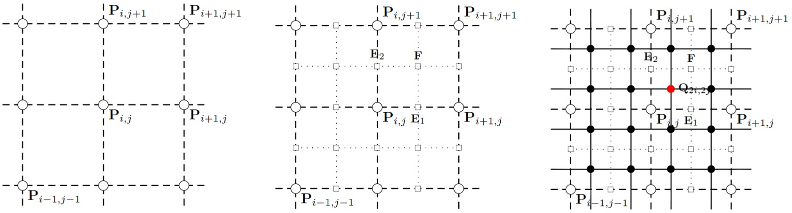
Non-uniform Biquadratic B-spline Surfaces



A horizonal knot interval d_i and a vertical knot interval e_j is assigned to each control point $\mathbf{P}_{i,j}$, as each control point corresponds to a biquadratic surface patch.

Non-uniform Biquadratic B-spline Subdivision

Refinement rules



Repeated averaging:

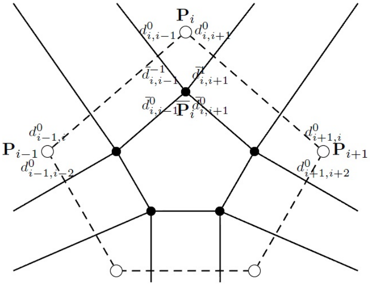
- Non-uniform linear subdivision:

$$\mathbf{E}_1 = \frac{d_{i+1}\mathbf{P}_{i,j} + d_i\mathbf{P}_{i+1,j}}{d_i + d_{i+1}}, \quad \mathbf{E}_2 = \frac{e_{j+1}\mathbf{P}_{i,j} + e_j\mathbf{P}_{i,j+1}}{e_j + e_{j+1}}$$

$$\mathbf{F} = \frac{e_{j+1}(d_{i+1}\mathbf{P}_{i,j} + d_i\mathbf{P}_{i+1,j}) + e_j(d_{i+1}\mathbf{P}_{i,j+1} + d_i\mathbf{P}_{i+1,j+1})}{(d_i + d_{i+1})(e_j + e_{j+1})}$$

- Dual averaging: $\mathbf{Q}_{2i,2j} = \frac{1}{4}(\mathbf{P}_{i,j} + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{F})$

Non-uniform Doo-Sabin Surfaces

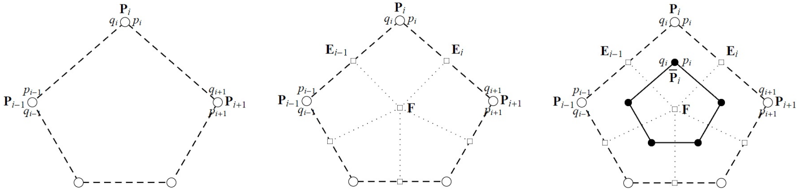


Each vertex is assigned a knot interval (possibly different) for each edge incident to it.
After subdivision, new knot intervals $\bar{d}^k_{ij}$ can be specified as follows:

$$\begin{array}{l} \bar{d}^0_{i,i+1} = \bar{d}^{-1}_{i,i-1} = d^0_{i,i+1} \\ \bar{d}^0_{i,i-1} = \bar{d}^{1}_{i,i+1} = d^0_{i,i-1} \end{array}.$$

Non-uniform Recursive Doo-Sabin Surfaces

Refinement rules



Repeated averaging:

- Non-uniform linear subdivision: $p_i = d^0_{i,i+1}$, $q_i = d^0_{i,i-1}$

$$\mathbf{E}_i = \frac{q_{i+1}\mathbf{P}_i + p_i\mathbf{P}_{i+1}}{p_i + q_{i+1}}$$

$$\mathbf{F} = \sum_{j=0}^{n-1} c_j \mathbf{P}_j$$

- Dual averaging: $\overline{\mathbf{P}}_i = \frac{1}{4}(\mathbf{P}_i + \mathbf{E}_{i-1} + \mathbf{E}_i + \mathbf{F})$

Non-uniform Recursive Doo-Sabin Surfaces

Coefficients c_j

Similarly to that in the Catmull-Clark variant of Doo-Sabin subdivision and non-uniform biquadratic B-spline subdivision, the face point $\mathbf{F}$ is the (weighted) centroid of the corresponding face and is the limit point corresponding to the center of the face.

$$\mathbf{F} = \sum_{j=0}^{n-1} c_j \mathbf{P}_j = \sum_{j=0}^{n-1} c_j \overline{\mathbf{P}}_j.$$

And,

$$\begin{aligned}\overline{\mathbf{P}}_i =& \frac{1}{4}\left(1 + c_i + \frac{q_{i+1}}{p_i + q_{i+1}} + \frac{p_{i-1}}{p_{i-1} + q_i}\right)\mathbf{P}_i + \frac{1}{4} \sum_{|i-j|>1} c_j \mathbf{P}_j \\ &+ \frac{1}{4}\left(c_{i-1} + \frac{q_i}{p_{i-1} + q_i}\right)\mathbf{P}_{i-1} + \frac{1}{4}\left(c_{i+1} + \frac{p_i}{p_i + q_{i+1}}\right)\mathbf{P}_{i+1}.\end{aligned}$$

Non-uniform Recursive Doo-Sabin Surfaces

Coefficients c_j

Combining the two equations, one obtains a system of linear equations with respect to $c_j, j = 0, \ldots, n-1$. Then we have

$$c_j = \frac{\alpha_j}{\sum_{k=0}^{n-1} \alpha_k},$$

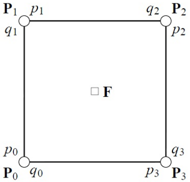
where

$$\alpha_j = \frac{1}{2}\left(\prod_{k=0}^{n-1} p_{j+k} + \prod_{k=0}^{n-1} q_{j-k}\right) + \sum_{m=1}^{n-1}\left(\prod_{k=1}^{m} q_{j+k} \prod_{k=m}^{n-1} p_{j+k}\right).$$

Here, indices are taken modulo n.

Non-uniform Recursive Doo-Sabin Surfaces

Quad case



$$\alpha_0 = \frac{p_0p_1p_2p_3 + q_0q_1q_2q_3}{2} + q_1p_1p_2p_3 + q_1q_2p_2p_3 + q_1q_2q_3p_3$$

$$\alpha_1 = \frac{p_0p_1p_2p_3 + q_0q_1q_2q_3}{2} + q_2p_2p_3p_0 + q_2q_3p_3p_0 + q_2q_3q_0p_0$$

$$\alpha_2 = \frac{p_0p_1p_2p_3 + q_0q_1q_2q_3}{2} + q_3p_3p_0p_1 + q_3q_0p_0p_1 + q_3q_0q_1p_1$$

$$\alpha_3 = \frac{p_0p_1p_2p_3 + q_0q_1q_2q_3}{2} + q_0p_0p_1p_2 + q_0q_1p_1p_2 + q_0q_1q_2p_2$$

Non-uniform Recursive Doo-Sabin Surfaces

A closer look at α_j

α_j is the sum of some products of the n knot intervals, which correspond to the n vertices respectively.

$$\alpha_j = \alpha_{j,j} + \sum_{l=0, l\neq j}^{n-1} \alpha_{j,l}$$
$$= \frac{1}{2}\left(\prod_{k=0}^{n-1} p_{j+k} + \prod_{k=0}^{n-1} q_{j-k}\right) + \sum_{m=1}^{n-1}\left(\prod_{k=1}^{m} q_{j+k} \prod_{k=m}^{n-1} p_{j+k}\right).$$

where

$$\alpha_{j,j} = \frac{1}{2}\left(\prod_{k=0}^{n-1} p_{j+k} + \prod_{k=0}^{n-1} q_{j-k}\right)$$

$$\alpha_{j,l} = \prod_{k=1}^{m} q_{j+k} \prod_{k=m}^{n-1} p_{j+k}, \quad m = (l-j) \mod n$$

Non-uniform Recursive Doo-Sabin Surfaces I

A closer look at α_j

$\alpha_{j,j} = \frac{1}{2}(\prod_{k=0}^{n-1} p_{j+k} + \prod_{k=0}^{n-1} q_{j-k})$ can be associated with $\mathbf{P}_j$ as follows. There are two paths of length n from $\mathbf{P}_j$ to itself.

- The clockwise path:

$$\mathbf{P}_j \to \mathbf{P}_{j+1} \to \cdots \to \mathbf{P}_{n-1} \to \mathbf{P}_0 \to \cdots \to \mathbf{P}_{j-1}$$

  then the product of the associated knot intervals is

$$\prod_{k=0}^{n-1} p_{j+k} = \prod_{k=0}^{n-1} p_k$$

Non-uniform Recursive Doo-Sabin Surfaces II

A closer look at α_j

- The counterclockwise path:

$$\mathbf{P}_j \rightarrow \mathbf{P}_{j-1} \rightarrow \cdots \rightarrow \mathbf{P}_0 \rightarrow \mathbf{P}_{n-1} \rightarrow \cdots \rightarrow \mathbf{P}_{j+1}$$

then the product of the associated knot intervals is

$$\prod_{k=0}^{n-1} q_{j-k} = \prod_{k=0}^{n-1} q_k$$

$\alpha_{j,j}$ is the average of the previous two products.

Non-uniform Recursive Doo-Sabin Surfaces I

A closer look at α_j

For $l \neq j$, $\alpha_{j,l} = \prod_{k=1}^{m} q_{j+k} \prod_{k=m}^{n-1} p_{j+k}$ can be associated with $\mathbf{P}_l$ in the following way. Here, $m = (l - j) \mod n$.

There exist one path of length $|l - j|$ and one path of length $n - |l - j|$ from $\mathbf{P}_l$ to $\mathbf{P}_j$.

If $l > j$, then we have

- The counterclockwise path of length $m = l - j$:

$$\mathbf{P}_l \rightarrow \mathbf{P}_{l-1} \rightarrow \cdots \rightarrow \mathbf{P}_{j+1} \rightarrow \mathbf{P}_j$$

  then the m associated knot intervals are

$$q_l, q_{l-1}, \ldots, q_{j+1},$$

  and their product is $\prod_{k=1}^{m} q_{j+k}$.

Non-uniform Recursive Doo-Sabin Surfaces II

A closer look at α_j

- The clockwise path of length $n - m = n - (l - j)$:

$$\mathbf{P}_l \to \mathbf{P}_{l+1} \to \cdots \to \mathbf{P}_{n-1} \to \mathbf{P}_0 \to \cdots \to \mathbf{P}_{j-1}$$

  then the $n - m$ associated knot intervals are

$$p_l, p_{l+1}, \ldots, p_{n-1}, p_0, \ldots, p_{j-1},$$

  and their product is $\prod_{k=m}^{n-1} p_{j+k}$. Here, indices are taken modulo n.

$\alpha_{j,l}$ is the product of the above two products.

For $l < j$, we have a similar explanation for $\alpha_{j,l}$.

Non-uniform Recursive Doo-Sabin Surfaces

Convergence

Theorem (Convergence)

The NURDS scheme is convergent at extraordinary points of arbitrary valence.

Corollary (Limit point rule)

For an n*-sided face, its face point (i.e. the weighted centroid) is the limit point of its associated extraordinary point.*

Non-uniform Recursive Doo-Sabin Surfaces

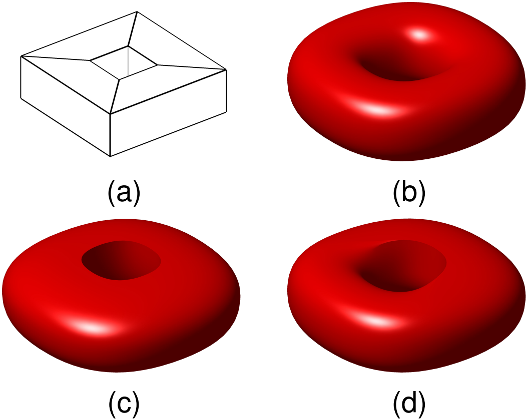
Continuity

NURDSes have stationary subdivision rules.

- We prove that the NURDS scheme is G^1 at vertices of valence 3 or 4.
- Because the subdivision matrix has no obvious symmetries, it is difficult to perform an eigenanalysis for extraordinary points with valence $n \geq 5$. Numerical experiments and examples show that limit surfaces are G^1 at these points.

Non-uniform Recursive Doo-Sabin Surfaces

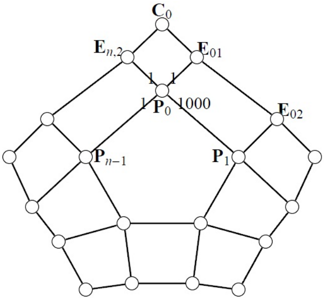
Examples



(a) (b) (c) (d)

(a) initial control mesh, (b) uniform biquadratic B-spline surface, (c) biquadratic NURBS surface with a crease, (d) NURDS with a dart.

NURDSes vs Quadratic NURSSes



Consider the configuration surrounding a type F face of valence n, and assume that $p_0 = 1000$ and all other knot intervals equal 1.

- For valence $3 \leq n \leq 30$, we construct subdivision matrices for quadratic NURSSes and NURDSes respectively, and then investigate eigenstructure and continuity.

NURDSes vs Quadratic NURSSes

Quadratic NURSSes

Concerning spectrum and continuity, we have the following results.

- For $3 \leq n \leq 30$, λ_0, λ_1 and λ_2 may be negative, while λ_3 is always positive.
- For $n \geq 15$, $|\lambda_0| > 1$, the subdivision process is divergent.
- For $3 \leq n \leq 14$, $\lambda_0 = 1 > |\lambda_1|$, the subdivision process is convergent.
- For $3 \leq n \leq 10$ and $n = 12$, $\lambda_0 = 1 > |\lambda_1| = |\lambda_2| > \lambda_3$, quadratic NURSSes are G^1 continuous at the extraordinary vertices.
- For $n = 13$ and 14, $\lambda_0 = 1 > |\lambda_1| > |\lambda_2| > \lambda_3$, quadratic NURSSes are G^1 continuous at the extraordinary vertices.
- For $n = 11$, $\lambda_0 = 1 > |\lambda_1| > \lambda_2 = \lambda_3$, quadratic NURSSes are only G^0 continuous at the extraordinary vertices.

NURDSes vs Quadratic NURSSes

Quadratic NURSSes

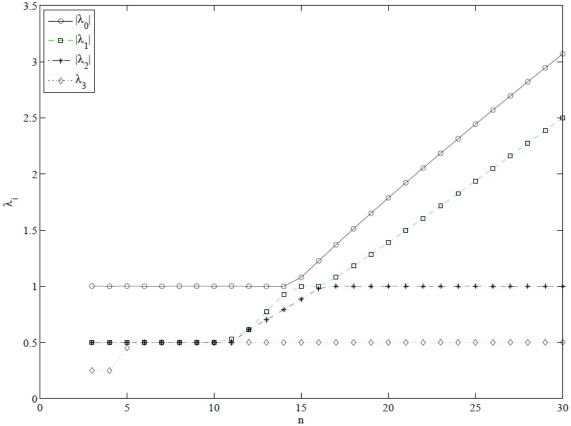


Figure: Absolute values of the first four eigenvalues for quadratic NURSSes for $3 \leq n \leq 30$.

NURDSes vs Quadratic NURSSes

NURDSes

Regarding spectrum and continuity, it follows that

- For $3 \leq n \leq 30$, $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are all positive.
- For $3 \leq n \leq 30$, $\lambda_0 = 1 > |\lambda_1|$, the subdivision process is convergent.
- For $3 \leq n \leq 30$, $\lambda_0 = 1 > \lambda_1 = \lambda_2 > \lambda_3$, NURDSes are G^1 continuous at the extraordinary vertices.

NURDSes vs Quadratic NURSSes

NURDSes

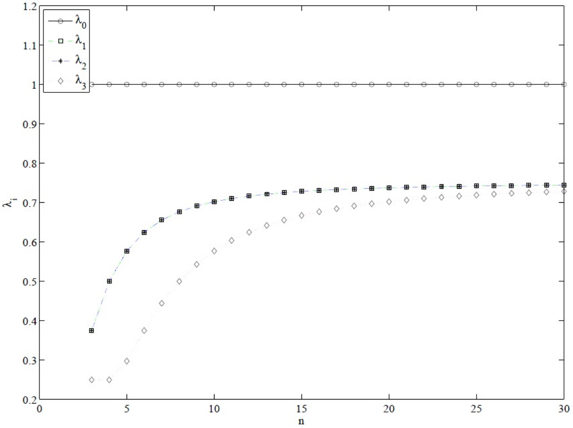


Figure: Values of the first four eigenvalues for NURDSes for $3 \leq n \leq 30$.

NURDSes vs Quadratic NURSSes

Both NURDSes and quadratic NURSSes are the subdivision surfaces that generalize non-uniform biquadratic B-spline surfaces to control grids of arbitrary topology.

Differences:

- NURDSes reduce to Catmull-Clark-variant Doo-Sabin surfaces whereas quadratic NURSSes degenerate to original Doo-Sabin surfaces.
- NURDSes are convergent at extraordinary points of arbitrary valence while quadratic NURSSes may diverge for valences larger than 12.
- NURDSes have closed form limit point rules whereas quadratic NURSSes do not.

Summary

- NURDSes are a generalization of Catmull-Clark-variant Doo-Sabin surfaces and biquadratic NURBS surfaces.
- NURDS refinement can be factored into non-uniform linear subdivision followed by dual averaging.
- NURDSes are convergent for arbitrary n-sided faces.
- NURDSes have closed form limit point rules.

- Future work:
  - Rigorous analysis for G^1 continuity for valence $n > 5$.
  - Boundary rules for open meshes.
  - Generalization of repeated averaging to higher degree cases, such as bicubic NURBS.

Thanks!