Estimating Error Bounds and Subdivision Depths for Loop Subdivision Surfaces

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Abstract

We derive a bound on the maximal distance between a Loop subdivision surface patch and its control mesh in terms of the maximum norm of the mixed second differences of the control points and a constant that depends only on the valence of the patch. A subdivision depth formula is also proposed.

Keywords: Loop subdivision surface, control mesh, error bound, subdivision depth

1 Introduction

A subdivision surface is defined as the limit of a finer and finer control mesh by subdividing the mesh recursively. The Loop subdivision surface generalizes the quartic three-directional box spline surface to triangular meshes of arbitrary topology [1].

Previous error estimation techniques for Loop subdivision surfaces can be classified into two classes.

One is the vertex based method [2, 3], which measures the distance between the vertices and their limit positions. Lanquetin et al. derived a wrong exponential bound and consequently a wrong subdivision depth formula [2]. Wang et al. proposed a proper exponential bound with an inefficient subdivision depth estimation technique [3]. As pointed out in [3], the vertex based bounds all suffer from one problem: they may be smaller than the actual distance in some cases.

The other is the patch based method [4, 5], which estimates the parametric distance between a Loop surface patch and its control mesh. Wu et al. presented an accurate error measure [4, 5]. But their estimate is dependent on recursive subdivision, thus can not be used to pre-compute the error bound after several steps of subdivision or predict the recursion depth within a user-specified tolerance.

Our technique belongs to the latter. Based on Stam's parametrization [6], we derive a conservative but safe bound on the maximum distance between a Loop surface patch and its control mesh. An efficient subdivision depth formula is also given.

2 Second Order Norm and Convergence Rate

Without loss of generality, we assume the initial triangular control mesh has been subdivided at least once, isolating the extraordinary vertices so that each face is a triangle and contains at most one extraordinary vertex.

2.1 Distance and Second Order Norm

Given a control mesh of a Loop subdivision surface $\tilde{\mathbf{S}}$, for each interior triangle face $\mathbf{F}$ in the control mesh, there is a corresponding surface patch $\mathbf{S}$ in the limit surface $\tilde{\mathbf{S}}$.

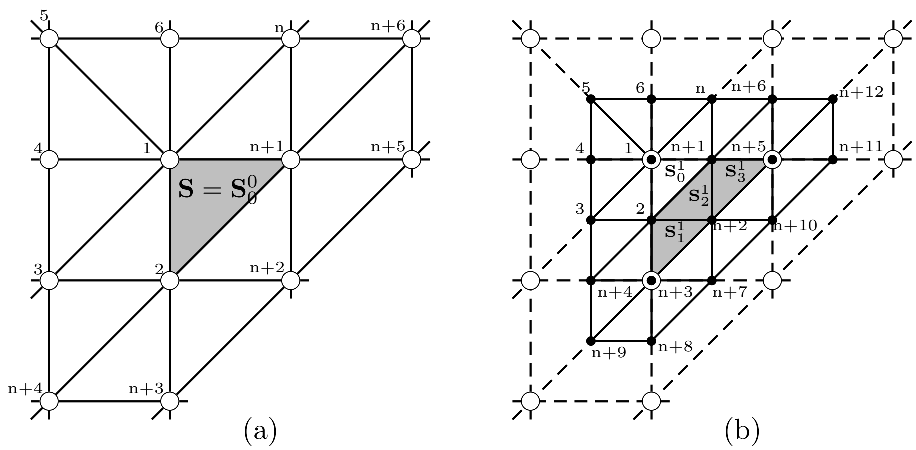

Figure 1: (a) Ordering of the control vertices of an extraordinary patch $\mathbf{S}$ of valence n. (b) Ordering of the new control vertices (*solid dots*) after one step of Loop subdivision.

The *distance* between a Loop patch $\mathbf{S}$ and the corresponding triangle $\mathbf{F}$ is defined as the maximum distance between $\mathbf{S}(v,w)$ and $\mathbf{F}(v,w)$, that is,

$$\max_{(v,w)\in\Omega} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| \ ,$$

where Ω is the unit triangle $\Omega = \{(v,w)|v \in [0,1] \text{ and } w \in [0,1-v]\}$, $\mathbf{S}(v,w)$ is the Stam's parametrization of $\mathbf{S}$ over Ω, and $\mathbf{F}(v,w)$ is the linear parametrization of $\mathbf{F}$ over Ω.

The control mesh Π of a Loop patch $\mathbf{S}$ consists of $n+6$ control vertices $\Pi = \{\mathbf{P}_i : i = 1, 2, \ldots, n+6\}$, where n is the valence of $\mathbf{F}$'s only extraordinary vertex (if has, otherwise $n = 6$) and called the *valence* of the patch $\mathbf{S}$ (see Figure 1(a)).

The *second order norm* of Π, denoted $M = M^0 = M_0^0$, is defined as the maximum norm of the following $n+9$ mixed second differences (MSDs) $\{\alpha_i : i \le i \le n+9\}$ of the $n+6$ vertices of Π:

$$\begin{aligned} M = \max\{ & \|\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{n+1} - \mathbf{P}_3\|, \\ & \{\|\mathbf{P}_1 + \mathbf{P}_i - \mathbf{P}_{i-1} - \mathbf{P}_{(i-1)\%n+2}\| : 3 \le i \le n+1\}, \\ & \|\mathbf{P}_2 + \mathbf{P}_{n+1} - \mathbf{P}_1 - \mathbf{P}_{n+2}\|, \|\mathbf{P}_2 + \mathbf{P}_{n+2} - \mathbf{P}_{n+1} - \mathbf{P}_{n+3}\|, \\ & \|\mathbf{P}_2 + \mathbf{P}_{n+3} - \mathbf{P}_{n+2} - \mathbf{P}_{n+4}\|, \|\mathbf{P}_2 + \mathbf{P}_{n+4} - \mathbf{P}_{n+3} - \mathbf{P}_3\|, \\ & \|\mathbf{P}_2 + \mathbf{P}_3 - \mathbf{P}_{n+4} - \mathbf{P}_1\|, \|\mathbf{P}_{n+1} + \mathbf{P}_n - \mathbf{P}_1 - \mathbf{P}_{n+6}\|, \\ & \|\mathbf{P}_{n+1} + \mathbf{P}_{n+6} - \mathbf{P}_n - \mathbf{P}_{n+5}\|, \|\mathbf{P}_{n+1} + \mathbf{P}_{n+5} - \mathbf{P}_{n+6} - \mathbf{P}_{n+2}\|, \\ & \|\mathbf{P}_{n+1} + \mathbf{P}_{n+2} - \mathbf{P}_{n+5} - \mathbf{P}_2\|\} \\ = \max\{ & \|\alpha_i\| : i = 1, \ldots, n+6\} \ . \end{aligned} \tag{1}$$

M is also called as the (level-0) second order norm of the patch $\mathbf{S}$. For a regular patch ($n = 6$), there are 15 mixed second differences.

Through subdivision we can generate $n+12$ new vertices $\mathbf{P}_i^1$, $i = 1, \ldots, n+12$ (see Figure 1(b)), which are called the level-1 control vertices of $\mathbf{S}$. All these level-1 control vertices compose $\mathbf{S}$'s level-1 control mesh $\Pi^1 = \{\mathbf{P}_i^1 : i = 1, 2, \ldots, n+12\}$. We use $\mathbf{P}_i^k$ and Π^k to represent the level-k control vertices and level-k control mesh of $\mathbf{S}$, respectively, after k steps of subdivision on Π.

The level-1 control vertices form four control vertex sets $\Pi_0^1, \Pi_1^1, \Pi_2^1$ and Π_3^1, corresponding to the control meshes of the subpatches $\mathbf{S}_0^1, \mathbf{S}_1^1, \mathbf{S}_2^1$ and $\mathbf{S}_3^1$, respectively (see Figure 1b), where $\Pi_0^1 = \{\mathbf{P}_i^1 : 1 \le i \le n+6\}$. The subpatch $\mathbf{S}_0^1$ is an extraordinary patch, but $\mathbf{S}_1^1, \mathbf{S}_2^1$ and $\mathbf{S}_3^1$ are regular triangular patches [6]. Following the definition in Euatioin (1), one can define the second order norms M_i^1 for $\mathbf{S}_i^1, i = 0, 1, 2, 3$, respectively. $M^1 = \max\{M_i^1 : i = 0, 1, 2, 3\}$ is defined as the second order norm of the level-1 control mesh Π^1. After k steps of subdivision on Π, one gets 4^k control point sets $\Pi_i^k : i = 0, 1, \ldots, 4^k - 1$ corresponding to the 4^k subpatches $\mathbf{S}_i^k : i = 0, 1, \ldots, 4^k - 1$ of $\mathbf{S}$, with $\mathbf{S}_0^k$ being the only level-k extraordinary patch (if $n \ne 6$). We denote M_i^k and M^k as the second order norms of Π_i^k and Π^k, respectively.

2.2 Convergence Rate

If second order norms M_0^{k+j} and M_0^j satisfy the following recurrence inequality

$$M_0^{k+j} \le r_j(n) M_0^k, \quad j \ge 0 \ , \tag{2}$$

where $r_j(n)$ is a constant which depends on n, the valence of the extraordinary vertex, and $r_0(n) \equiv 1$. We call $r_j(n)$ the *j-step convergence rate* of second order norm.

In the following, we estimate $r_j(n), j \geq 1$ by solving constrained minimization problems. Let $\alpha_i^k, i = 1, 2, \ldots, n+9$ be the MSDs of $\Pi_0^k, k \geq 0$ defined as in Equation (1). For each $l = 1, 2, \ldots, n+9$, we can express α_l^{k+1} as a linear combination of α_i^k:

$$\alpha_l^{k+1} = \sum_{i=1}^{n+9} x_i^l \alpha_i^k \ ,$$

where $x_i^l, i = 1, 2, \ldots, n+9$ are undetermined real coefficients. It follows that

$$\|\alpha_l^{k+1}\| \leq \sum_{i=1}^{n+9} \|x_i^l \alpha_i^k\| \leq \sum_{i=1}^{n+9} |x_i^l| \|\alpha_i^k\| \leq \sum_{i=1}^{n+9} |x_i^l| M_0^k \ .$$

Then we can bound $\|\alpha_l^{k+1}\|$ by $c_l(n) M_0^k$, where $c_l(n)$ is the solution of the following constrained minimization problem:

$$\begin{aligned} c_l(n) &= \min \sum_{i=1}^{n+9} |x_i^l| \ , \\ s.t. \quad & \sum_{i=1}^{n+9} x_i^l \alpha_i^k = \alpha_l^{k+1} \ . \end{aligned} \tag{3}$$

Since $M_0^{k+1} = \max\{\|\alpha_l^{k+1}\| : 1 \leq l \leq n+9\}$, we get an estimate for $r_1(n)$ as follows:

$$r_1(n) = \max_{1 \leq l \leq n+9} c_l(n) \ .$$

By symmetry, we only need to solve at most four constrained minimization problems corresponding to $\alpha_1^{k+1} = \mathbf{P}_1^{k+1} + \mathbf{P}_2^{k+1} - \mathbf{P}_{n+1}^{k+1} - \mathbf{P}_3^{k+1}$, $\alpha_{n+1}^{k+1} = \mathbf{P}_2 + \mathbf{P}_{n+1} - \mathbf{P}_1 - \mathbf{P}_{n+2}$, $\alpha_{n+2}^{k+1} = \mathbf{P}_2 + \mathbf{P}_{n+2} - \mathbf{P}_{n+1} - \mathbf{P}_{n+3}$, and $\alpha_{n+3}^{k+1} = \mathbf{P}_2 + \mathbf{P}_{n+3} - \mathbf{P}_{n+2} - \mathbf{P}_{n+4}$, respectively. Since $\mathbf{P}_1^{k+1}$ is the extraordinary vertex, it is not surprising to find out that $c_1(n)$ is the maximum for $n > 3$. The special case is $r_1(3) = c_4(3) = 0.4375 > c_1(3) = 0.3125$. Then it follows that

$$r_1(n) = c_1(n), \ n > 3 \ .$$

Similarly, we can estimate $r_j(n), \ n \geq 3, \ j > 1$ by solving only one constrained minimization problem (3) with α_1^{k+1} replaced by α_1^{k+j}. Table 1 shows the convergence rates of the second order norm for $3 \leq n \leq 10$. The convergence rate $r_j(6) = (\frac{1}{4})^j$ is sharp for regular patches.

Table 1: Convergence rates $r_i(n),\ i = 1, 2, 3$

n	3	4	5	6	7	8	9	10
$r_1(n)$	0.437500	0.382813	0.540907	0.250000	0.674025	0.705136	0.726355	0.741711
$r_2(n)$	0.082031	0.142700	0.258367	0.062500	0.372582	0.402608	0.424000	0.439960
$r_3(n)$	0.020752	0.053148	0.118899	0.015625	0.197695	0.219995	0.236377	0.248872

3 Distance Bound

3.1 Regular Patch

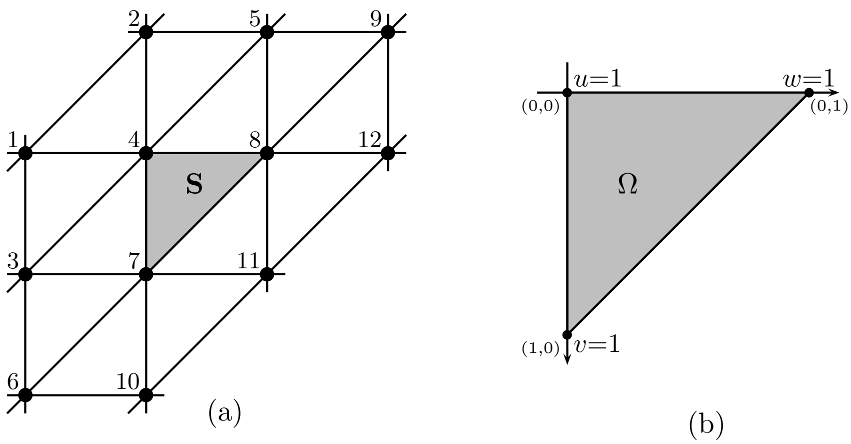

Figure 2: (a) Control vertices of a quartic box spline patch and their ordering. (b) Parameter domain $\Omega = \{(v, w)|v \in [0, 1] \text{ and } w \in [0, 1 - v]\}$ with the third parameter $u = 1 - v - w$.

If $\mathbf{S}$ is a regular Loop patch, then $\mathbf{S}(v, w)$ can be expressed as a quartic box spline patch defined over the unit triangle Ω with control vertices $\mathbf{p}_i, 1 \leq i \leq 12$ (see Figure 2) as follows:

$$\mathbf{S}(v, w) = \sum_{i=1}^{12} \mathbf{p}_i N_i(v, w) \quad , \tag{4}$$

where $N_i(v, w), 1 \leq i \leq 12$ are the quartic box spline basis functions. The expressions of $N_i(v, w)$ refer to [6].

$\mathbf{S}$ is also a quartic triangular Bézier patch, thus $\mathbf{S}(v, w)$ can be written in terms of Bernstein polynomials [8]:

$$\mathbf{S}(v, w) = \sum_{i=1}^{15} \mathbf{b}_i B_i(v, w) \quad , \tag{5}$$

where $\mathbf{b}_i, 1 \leq i \leq 15$ are the Bézier points of $\mathbf{S}$, and $B_i(v, w), 1 \leq i \leq 15$ are the quartic Bernstein polynomials (see Figure 3). The correspondence between the standard representation $B^4_{ijk}(u,v,w) = \frac{4!}{i!j!k!}u^i v^j w^k, i,j,k \geq 0, i+j+k=4$ and $B_i(v,w), 1 \leq i \leq 15$ is $(i,j,k) \mapsto \frac{(j+k)(j+k+1)}{2} + k + 1$.

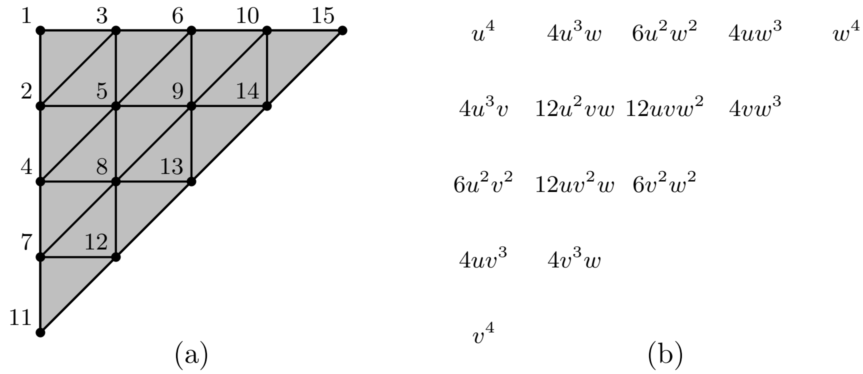

Figure 3: (a) Ordering of the Bézier points of a quartic triangular Bézier patch. (b) Quartic Bernstein polynomials corresponding to the Bézier points.

The 15×12 matrix T which converts from the 12 control vertices to the 15 Bézier points can be computed with the algorithm developed in [7]:

$$T = \frac{1}{24}\begin{bmatrix} 2 & 2 & 2 & 12 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 12 & 1 & 0 & 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 12 & 3 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 & 0 & 8 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 & 1 & 0 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 4 & 0 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 1 & 12 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 & 10 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 6 & 10 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 3 & 0 & 3 & 12 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 12 & 4 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 8 & 8 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 4 & 12 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 \end{bmatrix}$$

The linear parametrization of the center triangle $\mathbf{F} = \{\mathbf{p}_4, \mathbf{p}_7, \mathbf{p}_8\}$ (see Figure 4(a)) is

$$\mathbf{F}(v,w) = u\mathbf{p}_4 + v\mathbf{p}_7 + w\mathbf{p}_8 \ ,$$

where $u = 1 - v - w$. By the linear precision property of the Bernstein polynomials [8], we can express $\mathbf{F}(v,w)$ as the following quartic Bézier form:

$$\mathbf{F}(v,w) = \sum_{i=1}^{15} \overline{\mathbf{b}}_i B_i(v,w) \ , \tag{6}$$

where $\overline{\mathbf{b}}_{\frac{(j+k)(j+k+1)}{2}+k+1} = \mathbf{F}(\frac{j}{4}, \frac{k}{4}), j,k \geq 0, 0 \leq j+k \leq 4$ are the Bézier points. It is obvious that $\mathbf{p}_4 = \overline{\mathbf{b}}_1, \mathbf{p}_7 = \overline{\mathbf{b}}_{11}, \mathbf{p}_8 = \overline{\mathbf{b}}_{15}$.

Hence, for $(v,w) \in \Omega$, it follows that

$$\begin{aligned} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| &= \left\| \sum_{i=1}^{15} (\mathbf{b}_i - \overline{\mathbf{b}}_i) B_i(v,w) \right\| \\ &\leq \sum_{i=1}^{15} \|\mathbf{b}_i - \overline{\mathbf{b}}_i\| B_i(v,w) \ . \end{aligned} \tag{7}$$

We can bound $\|\mathbf{b}_i - \overline{\mathbf{b}}_i\|$ in terms of M as $\|\mathbf{b}_i - \overline{\mathbf{b}}_i\| \leq \delta_i M$. Here

$$\begin{aligned} &\delta_1 = \delta_{11} = \delta_{15} = \frac{1}{2} \ , \\ &\delta_2 = \delta_3 = \delta_7 = \delta_{10} = \delta_{12} = \delta_{14} = \frac{1}{4} \ , \\ &\delta_4 = \delta_6 = \delta_{13} = \frac{1}{6} \ , \\ &\delta_5 = \delta_8 = \delta_9 = \frac{1}{12} \ . \end{aligned}$$

It is easy to know that $\sum_{i=1}^{15} \delta_i B_i(v,w)$ reaches its maximum at the three corners of Ω, that is,

$$\max_{(v,w)\in\Omega} \sum_{i=1}^{15} \delta_i B_i(v,w) = \frac{1}{2} \ .$$

Then we have a bound on the maximal distance between $\mathbf{S}(v,w)$ and $\mathbf{F}(v,w)$ as stated in the following theorem:

Theorem 3.1 *The distance between a regular Loop patch* $\mathbf{S}$ *and the corresponding center triangle* $\mathbf{F}$ *is bounded by*

$$\max_{(v,w)\in\Omega} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| \leq \frac{1}{2} M \ .$$

3.2 Extraordinary Patch

An extraordinary Loop patch $\mathbf{S}$ can be partitioned into an infinite sequence of regular triangular patches $\{\mathbf{S}_m^k\}, k \geq 1, m = 1,2,3$. If we partition the unit triangle Ω into an infinite set of tiles $\{\Omega_m^k\}, k \geq 1, m = 1,2,3$ accordingly, (see Figure 4), with

$$\begin{aligned} \Omega_1^k &= \{(v,w) \,|\, v \in [2^{-k}, 2^{-k+1}] \text{ and } w \in [0, 2^{-k+1} - v]\} \ , \\ \Omega_2^k &= \{(v,w) \,|\, v \in [0, 2^{-k}] \text{ and } w \in [2^{-k} - v, 2^{-k}]\} \ , \\ \Omega_3^k &= \{(v,w) \,|\, v \in [0, 2^{-k}] \text{ and } w \in [2^{-k}, 2^{-k+1} - v]\} \ . \end{aligned}$$

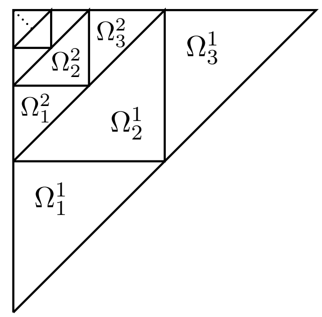

Figure 4: Partition of the parameter domain Ω.

Each tile Ω_m^k corresponds to a box spline patch $\mathbf{S}_m^k$. And we denote the parameter space corresponding to the extraordinary subpatch $\mathbf{S}_0^k$ by

$$\Omega_0^k = \{(v,w)\,|\, v \in [0, 2^{-k}] \text{ and } w \in [0, 2^{-k} - v]\} \ .$$

Let $\mathbf{F}(v,w)$, $\mathbf{F}_m^k(v,w)$ and $\mathbf{F}_0^k(v,w)$ be the linear parametrization of the center faces of the control meshes of $\mathbf{S}$, $\mathbf{S}_m^k$ and $\mathbf{S}_0^k$, respectively. Using the triangle inequality, for $(v,w) \in \Omega_m^k$, $m = 1,2,3$, the distance between an extraordinary Loop patch $\mathbf{S}(v,w)$ and the corresponding triangle $\mathbf{F}(v,w)$ can be bounded as

$$\begin{aligned} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| &\le \|\mathbf{S}_m^k(v,w) - \mathbf{F}_m^k(v,w)\| \\ &+ \|\mathbf{F}_m^k(v,w) - \mathbf{F}_0^{k-1}(v,w)\| + \sum_{i=0}^{k-2} \|\mathbf{F}_0^{i+1}(v,w) - \mathbf{F}_0^i(v,w)\| \end{aligned} \ . \quad (8)$$

With a proof analogous to the one of Lemma 3 in [9], we have the following two lemmas.

Lemma 3.2 *If $(v,w) \in \Omega_m^k$, $m = 1,2,3$, then*

$$\left\|\mathbf{F}_m^k(v,w) - \mathbf{F}_0^{k-1}(v,w)\right\| \le \begin{cases} \frac{3}{8} M_0^{k-1}, & m{=}1,3; \\ \frac{1}{8} M_0^{k-1}, & m{=}2. \end{cases} \ ,$$

where M_0^{k-1} is the second order norm of $\mathbf{S}_0^{k-1}$.

Lemma 3.3 *If $(v,w) \in \Omega_m^k$, then for any $0 \le i \le k-2$ we have*

$$\left\|\mathbf{F}_0^{i+1}(v,w) - \mathbf{F}_0^i(v,w)\right\| \le \omega(n) M_0^i \ ,$$

where $\omega(n) = \frac{5}{8} - \frac{(3+2\cos(2\pi/n))^2}{64}$, and M_0^i is the second order norm of $\mathbf{S}_0^i$ and $\mathbf{F}_0^0 = \mathbf{F}$.

It follows that if $(v, w) \in \Omega_m^k, m = 1, 3$,

$$\begin{aligned} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| &\leq \frac{1}{2}M_m^k + \frac{3}{8}M_0^{k-1} + \omega(n)\sum_{i=0}^{k-2} M_0^i \\ &\leq \frac{1}{2}M_m^k + \frac{3}{8}M_0^{k-1} + \omega(n)\sum_{i=0}^{k-2} r_i(n)M^0 \end{aligned} \tag{9}$$

Here, $r_i(n)$ is the i-step convergence rate of second order norm. Let $m \to \infty$ in Equation (9), we get

$$\|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| \leq \omega(n)\sum_{i=0}^{\infty} r_i(n)M^0 \ .$$

Because $\{\Omega_m^k\}, k \geq 1,\ m = 1, 2, 3$, form a partition of Ω, we have the following theorem on the maximal distance between $\mathbf{S}(v, w)$ and $\mathbf{F}(v, w)$, $(v, w) \in \Omega$:

Theorem 3.4 *The distance between an extraordinary Loop patch* $\mathbf{S}$ *and the corresponding triangle* $\mathbf{F}$ *is bounded by*

$$\max_{(v,w)\in\Omega} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| \leq C_\infty(n)M^0 \ ,$$

where

$$C_\infty(n) = \omega(n)\sum_{i=0}^{\infty} r_i(n) \ ,$$

and $M^0 = M$ *is the second order norm of* $\mathbf{S}$.

For a regular patch with $n = 6$, we have $C_\infty(6) = \omega(6)/(1 - r_1(6)) = (3/8)/(1 - 1/4) = 1/2$. The result in Theorem 3.1 is obtained again. However, there are no explicit expressions of $r_i(n)$ for general n, we have the following practical corollary for error estimation.

Corollary 3.5 *The distance between an extraordinary Loop patch* $\mathbf{S}$ *and the corresponding triangle* $\mathbf{F}$ *is bounded as*

$$\max_{(v,w)\in\Omega} \|\mathbf{S}(v,w) - \mathbf{F}(v,w)\| \leq C_\lambda(n)M^0, \quad \lambda \geq 1 \ ,$$

where

$$C_\lambda(n) = \omega(n)\frac{\sum_{i=0}^{\lambda-1} r_i(n)}{1 - r_\lambda(n)} \ ,$$

and $M^0 = M$ *is the second order norm of* $\mathbf{S}$.

Table 2 gives the numerical results of the bound constants $C_\lambda(n)\,(\lambda = 1, 2, 3)$.

Table 2: Comparison of $C_\lambda(n)$ $(\lambda = 1, 2, 3)$

n	3	4	5	6	7	8	9	10
$C_1(n)$	1.000000	0.784810	0.915862	0.5	1.052763	1.087084	1.111167	1.129655
$C_2(n)$	0.880851	0.781290	0.873611	0.5	0.915630	0.914924	0.911327	0.907421
$C_3(n)$	0.872850	0.780397	0.858623	0.5	0.875407	0.866176	0.856245	0.847476

4 Subdivision Depth Estimation

Because the distance between a level-k control mesh and the surface patch $\mathbf{S}$ is dominated by the distance between the level-k extraordinary subpatch and its corresponding control mesh, which, according to Corollary 3.5, is

$$\|\mathbf{S}(v, w) - \mathbf{F}^k(v, w)\| \le C_\lambda(n) M^k \ ,$$

where M^k is the second order norm of $\mathbf{S}$'s level-k control mesh. Assume $\epsilon > 0$ and $k = \lambda l_j + j, 0 \le j \le \lambda - 1$, let

$$\begin{aligned}\|\mathbf{S}(v, w) - \mathbf{F}^k(v, w)\| &\le C_\lambda(n) r_k(n) M^0 \\ &\le C_\lambda(n) (r_\lambda(n))^{l_j} r_j(n) M^0 < \epsilon \ ,\end{aligned}$$

then it follows that $l_j \ge \left\lceil \log_{\frac{1}{r_\lambda(n)}} \left(\frac{r_j(n) C_\lambda(n) M^0}{\epsilon} \right) \right\rceil$. Consequently, we have the following subdivision depth estimation formula for extraordinary Loop patches.

Theorem 4.1 *Given an extraordinary Loop patch* $\mathbf{S}$ *and an error tolerance* $\epsilon > 0$*, after*

$$k = \min_{0 \le j \le \lambda - 1} \lambda l_j + j$$

steps of subdivision on the control mesh of $\mathbf{S}$*, the distance between* $\mathbf{S}$ *and its level-*k *control mesh is smaller than* ϵ*. Here,*

$$l_j = \left\lceil \log_{\frac{1}{r_\lambda(n)}} \left(\frac{r_j(n) C_\lambda(n) M^0}{\epsilon} \right) \right\rceil, \quad 0 \le j \le \lambda - 1, \lambda \ge 1 \ ,$$

where $r_j(n)$ *and* $C_\lambda(n)$ *are the same as in Corollary 3.5,* $M^0 = M$ *is the second order norm of* $\mathbf{S}$*.*

In particular, for a regular patch, $k = \left\lceil \log_4 \left(\frac{M^0}{2\epsilon} \right) \right\rceil$.

Assume the second order norm $M^0 = 1$, and the error tolerance $\epsilon = 0.01$. Table 3 shows the results of subdivision depths computed with different $\lambda = 1, 2, 3$.

Table 3: Comparison of subdivision depthes, $\lambda = 1, 2, 3$

n	3	4	5	6	7	8	9	10
$C_1(n)$	6	5	8	3	12	14	15	16
$C_2(n)$	4	5	7	3	10	10	11	12
$C_3(n)$	4	5	7	3	9	9	10	10

References

[1] Loop, C.T.: Smooth Subdivision Surfaces Based on Triangles. M.S. Thesis, Department of Mathematics, University of Utah (1987)

[2] Lanquetin, S., and Neveu, M.: A priori computation of the number of surface subdivision levels. In: Proceedings of International Conference on Computer Graphics and Vision 2004 (Graphicon'2004) (2004) 87–94

[3] Wang, H., Sun, H., and Qin, K.: Estimating recursion depth for Loop subdivision. International Journal of CAD/CAM **4:1** (2004) 11–18

[4] Wu, X., and Peters, J.: Interference detection for subdivision surfaces. Computer Graphics Forum, Eurographics 2004 **23:3** (2004) 577–585

[5] Wu, X., and Peters, J.: An Accurate Error Measure for Adaptive Subdivision Surfaces. In: Proceedings of International Conference on Shape Modeling and Applications 2005 (SMI'05) (2005) 51–56

[6] Stam, J.: Evaluation of Loop Subdivision Surfaces. In: Subdivision for Modeling and Anamation, SIGGRAPH 1999 Course Notes (1999)

[7] Lai, M.J.: Fortran Subroutines For B-Nets of Box Splines on Three- and Four-Directional Meshes. Numerical Algorithms, **2** (1992) 33–38

[8] Farin, G.: Curves and Surfaces for CAGD — A Practical Guide, 5th Edition. Morgan Kaufmann Publishers, San Francisco (2002)

[9] Huang, Z., and Wang, G.: Improved error estimate for extraordinary Catmull-Clark subdivision surface patches. The Visual Computer, **23:12** (2007) 1005–1014