高等数学复习(版本：3.14) zjzheng@mail.ustc.edu.cn整理：2006-9-19 习题课：2006-9-23

提纲：

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软件：

Matlab：长于数值计算，尽量采用矩阵运算，减少 for 循环

《高等应用数学问题的 Matlab 求解》薛定宇、陈阳泉，清华大学出版社 2004

Mathematica：长于符号运算，可以得到许多标准数学函数，Series[f,{x,0,5}]; FunctionExpand[f]

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1. 符号书写及其含义：o, O;　σ, δ;　χ, x;　μ, u;　$a, 2, \pi, \alpha, \partial$

符号	名称	TeX	用法	
$\equiv$	恒等于	\equiv	定义	
$=$	等于	=	赋值	
$\approx$ $\doteq$	约等于 近似于	\approx \doteq	近似	有效数字，P244 数量级 $3\cdot10^{n-1} < A \le 3\cdot10^{n}$
$\sim$	渐近于	\sim	渐近	
$\propto$	正比于	\propto	量级	
$\to$	趋近于	\to	趋近	

2. 数、函数及其性质：

A. 复数：求解形同 $x^2+1=0$ 一类的方程遇到的问题

虚数：$i=\sqrt{-1}$，复数：$z=x+iy$，共轭复数：$\overline{z}=z^*=x-iy$

复数的模：$|z| = \sqrt{z\,\bar{z}} = \sqrt{x^2+y^2}$，复数的辐角：$\operatorname{Arg} z = \arg z + 2n\pi,\quad n\in\mathbb{Z}$

复数的辐角主值：$\arg z = \varphi \in (-\pi,\pi),\quad \tan\varphi = y/x$

B. 常用函数

i. 指数函数和对数函数

$e^{ix} = \cos x + i\sin x$ (1748 年，复数的欧拉公式) $e^x = \cosh x + \sinh x$

$\ln z = \ln|z| + i\arg z$，$\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z$

特别地，$e^{i\pi}+1=0$ (欧拉公式)，$i^i = e^{i\operatorname{Ln} i} = e^{i(\ln 1 + i\operatorname{Arg} i)} = e^{-\pi/2-2n\pi},\quad n\in\mathbb{Z}$

ii. 三角函数：$\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$ $\quad\cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right)$

iii. 双曲函数：$(\sinh x)' = \cosh x,\quad (\cosh x)' = \sinh x,\quad (\tanh x)' = 1/(\cosh x)^2$

$\sinh x = \operatorname{sh} x = \frac{1}{2}\left(e^x - e^{-x}\right)$ $\qquad \operatorname{arsinh} = \sinh^{-1} x = \operatorname{sh}^{-1} x = \ln(x+\sqrt{1+x^2})$

$\cosh x = \operatorname{ch} x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\qquad \operatorname{arcosh} = \cosh^{-1} x = \operatorname{ch}^{-1} x = \ln(x\pm\sqrt{x^2-1}),\quad x\ge 1$

$\tanh x = \operatorname{th} x = \operatorname{sh} x/\operatorname{ch} x$ $\qquad \operatorname{artanh} = \tanh^{-1} x = \operatorname{th}^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x},\quad |x|<1$

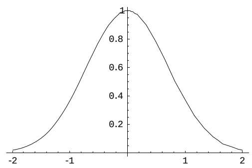
iv. Fresnel 积分：

$$\mathrm{S}(x) = \int_0^x \sin\frac{\pi}{2}t^2 dt;\quad \mathrm{C}(x) = \int_0^x \cos\frac{\pi}{2}t^2 dt;\quad \mathrm{C}(x) + i\,\mathrm{S}(x) = \int_0^x \exp(i\frac{\pi}{2}t^2)dt$$

特别地，$\mathrm{S}(\infty) = \mathrm{C}(\infty) = 1/2$

v. 误差函数和补余误差函数：

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt;\quad \operatorname{erfc}(x) = 1-\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$



特别地，$\operatorname{erf}(\infty) = 1;\quad \operatorname{erfc}(\infty) = 0$

vi. Euler 函数

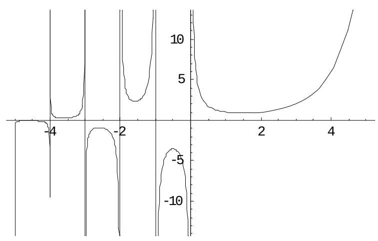
$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt;\quad \mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

$$\Gamma(x+1) = x\Gamma(x), x>0;\quad \Gamma(n+1) = n!;\quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x};$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2});\quad \Gamma(1/2) = \sqrt{\pi};\quad \int_0^{\pi/2}\sin^n x\cos^m x dx = \frac{1}{2}\mathrm{B}(\frac{n+1}{2},\frac{m+1}{2})$$

C. 函数的性质

i. 函数的极限：L'Hospital 法则



若 $\lim_{x\to x_0} f(x)=0, \lim_{x\to x_0} g(x)=0$，则 $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$

若 $\lim_{x\to x_0} f(x)=\infty, \lim_{x\to x_0} g(x)=\infty$，则 $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$

例如：$\lim_{x\to\pm\infty}(1+1/x)^x=\lim_{x\to\pm\infty}e^{x\ln(1+1/x)}=\lim_{t\to 0}e^{\frac{1}{t}\ln(1+t)}=e$

(P.92)对于有限的 m 和 $N>>m$，有

$$\left(1-m^2/N^2\right)^{N/2}=\left[\left(1-m^2/N^2\right)^{-N^2/m^2}\right]^{-m^2/2N}=e^{-m^2/2N}$$

$$\left(1\pm m/N\right)^{\pm m/2}=\left[\left(1\pm m/N\right)^{\pm N/m}\right]^{m^2/2N}=e^{m^2/2N}$$

ii. 函数 $f(x)$ 的极值：

极大值 $\begin{cases} f'(x)=0 \\ f''(x)<0 \end{cases}$；极小值 $\begin{cases} f'(x)=0 \\ f''(x)>0 \end{cases}$

iii. $\begin{cases} z=f(x,y) \\ \Phi(x,y)=0 \end{cases}$ 的条件极值：

构造辅助函数 $F(x,y)=f(x,y)+\lambda\Phi(x,y)$

求解 $\begin{cases} \partial F/\partial x=0 \\ \partial F/\partial y=0 \end{cases} \Rightarrow \begin{cases} \partial f/\partial x+\lambda\,\partial\Phi/\partial x=0 \\ \partial f/\partial y+\lambda\,\partial\Phi/\partial y=0 \end{cases}$，其中 λ 为 Lagrange 因子

D. 函数之间的关系（注意规范符号书写）

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\begin{cases} 0 & f(x)=o(g(x)),\ x\to x_0 & \text{远小于} \\ A & f(x)=O(g(x)),\ x\to x_0 & \text{同量阶} \\ 1 & f(x)\sim g(x),\ x\to x_0 & \text{等价量} \\ \infty & g(x)=o(f(x)),\ x\to x_0 & \text{远大于} \end{cases}$$

E. 函数的展开

i. 二项式展开：$(a+b)^n=\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}a^{n-m}b^m$

ii. Maclaurin 展开：$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$

iii. Taylor 展开：$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

渐近展开式

$$f(x,\varepsilon)=a_0(x)+a_1(x)\delta(\varepsilon)+a_2(x)\delta^2(\varepsilon)+a_3(x)\delta^3(\varepsilon)+\cdots,\quad |\delta(\varepsilon)|<<1$$

$$a_0(x)=f(x,0);\quad a_1(x)=\frac{1}{1!}\lim_{\varepsilon\to 0}\frac{1}{\delta'}f_\varepsilon(x,\varepsilon)$$

$$a_2(x)=\frac{1}{2!}\lim_{\varepsilon\to 0}\frac{1}{\delta'}\frac{d}{d\varepsilon}\left(\frac{f_\varepsilon(x,\varepsilon)}{\delta'}\right)=\frac{1}{2!}\lim_{\varepsilon\to 0}\frac{1}{\delta'}\left(\frac{1}{\delta'}f_{\varepsilon\varepsilon}(x,\varepsilon)-\frac{\delta''}{(\delta')^2}f_\varepsilon(x,\varepsilon)\right)$$

…

e.g. $\delta(\varepsilon)=\varepsilon,\quad a_0(x)=f(x,0);\quad a_1(x)=f_\varepsilon(x,0);a_2(x)=\frac{1}{2!}f_{\varepsilon\varepsilon}(x,0);\cdots$

渐近级数展开：

$$f(x(\varepsilon),\varepsilon)=a_0+a_1\varepsilon+a_2\varepsilon^2+a_3\varepsilon^3+\cdots$$

$$a_0=f(x(0),0)=f^{(0)}$$

$$a_1(x)=\frac{1}{1!}\lim_{\varepsilon\to 0}\left[f_x(x(\varepsilon),\varepsilon)x'(\varepsilon)+f_\varepsilon(x(\varepsilon),\varepsilon)\right]=f_x^{(0)}x'(0)+f_\varepsilon^{(0)}$$

$$a_2(x)=\frac{1}{2!}\lim_{\varepsilon\to 0}\left[\begin{array}{l}\left(f_{xx}(x(\varepsilon),\varepsilon)x'(\varepsilon)+f_{x\varepsilon}(x(\varepsilon),\varepsilon)\right)x'(\varepsilon)+f_x(x(\varepsilon),\varepsilon)x''(\varepsilon)\\+f_{\varepsilon x}(x(\varepsilon),\varepsilon)x'(\varepsilon)+f_{\varepsilon\varepsilon}(x(\varepsilon),\varepsilon)\end{array}\right]$$

$$=\frac{1}{2!}\left[f_{xx}^{(0)}\left(x'(0)\right)^2+2f_{x\varepsilon}^{(0)}x'(0)+f_x^{(0)}x''(0)+f_{\varepsilon\varepsilon}^{(0)}\right]$$

…

P53

$$f(x(\varepsilon),y(\varepsilon),\varepsilon)=a_0+a_1\varepsilon+a_2\varepsilon^2+a_3\varepsilon^3+\cdots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+o(x^n),\ |x|<<1$$ **(Maclaurin)**

$$e^x=e^{x_0}\sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}=e^{x_0}[1+(x-x_0)+\frac{1}{2!}(x-x_0)^2+\frac{1}{3!}(x-x_0)^3+\cdots]$$ **(Taylor)**

$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n),\ |x|<<1$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots,\ |x|<1$$

$$\sqrt{1+x}=1+\frac{x}{2}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}-\cdots,\ |x|\le 1$$

$$\sin x=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}-\cdots$$

$$\cos x = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m} = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\cdots$$

3. 微分方程：$F(x,y,y',...,y^{(n)})=0$

i. 可分离变量方程：

$$f_1(x)g_1(y)dx+f_2(x)g_2(y)dy=0 \quad\Rightarrow\quad \int\frac{f_1(x)}{f_2(x)}dx+\int\frac{g_1(y)}{g_2(y)}dy=C$$

ii. 齐次方程：

- 一阶：$\dfrac{dy}{dx}=F(\dfrac{y}{x}) \quad\Rightarrow\quad u=y/x,\quad \big(u-F(u)\big)dx+xdu=0$，可分离变量
- 二阶：$y''+p(x)y'+q(x)y=0 \Rightarrow y(x)=c_1y_1(x)+c_2y_1(x)\displaystyle\int\frac{e^{-\int p(t)dt}}{y_1^2(x)}dx$
- 二阶常系数：$y''+py'+qy=0 \Rightarrow k^2+pk+q=0,\quad y(x)=c_1e^{k_1x}+c_2e^{k_2x}$

iii. 线性方程：

- 一阶：$\dfrac{dy}{dx}+p(x)y=q(x) \quad\Rightarrow\quad y=e^{-\int p(x)dx}\left[\displaystyle\int q(x)e^{\int p(x)dx}dx+C\right]$

常数变易法：线性齐次方程 $\dfrac{dy}{dx}+p(x)y=0$ 的解 $y=Ae^{-\int p(x)dx}$，线性方程的解 $y=A(x)e^{-\int p(x)dx}$ 代入方程可得 $A(x)=\int q(x)e^{\int p(x)dx}dx+C$

- 二阶：$y''+p(x)y'+q(x)y=f(x) \Rightarrow y(x)=c_1(x)y_1(x)+c_2(x)y_2(x)$，常数变易法，$y'(x)=c_1'(x)y_1(x)+c_1(x)y_1'(x)+c_2'(x)y_2(x)+c_2(x)y_2'(x)$，为了在继续求微商时不出现系数函数的高阶微商，令 $c_1'(x)y_1(x)+c_2'(x)y_2(x)=0$（限制条件），

$y'(x)=c_1(x)y_1'(x)+c_2(x)y_2'(x)$，

$y''(x)=c_1'(x)y_1'(x)+c_1(x)y_1''(x)+c_2'(x)y_2'(x)+c_2(x)y_2''(x)$，代入微分方程：

$c_1(x)\big(y_1''(x)+p(x)y_1'(x)+q(x)y_1(x)\big)$

$+c_2(x)\big(y_2''(x)+p(x)y_2'(x)+q(x)y_2(x)\big)$，则 $c_1'(x)y_1'(x)+c_2'(x)y_2'(x)=f(x)$

$+c_1'(x)y_1'(x)+c_2'(x)y_2'(x)=f(x)$

例如：$\begin{cases} y''+y=x^{-1} \\ y(\pi)=0,\quad y'(\pi)=0 \end{cases}$，常数变易法：$y(x)=c_1(x)\sin x+c_2(x)\cos x$

$$\begin{cases} c_1'(x)\sin x + c_2'(x)\cos x = 0 \\ c_1'(x)\cos x - c_2'(x)\sin x = x^{-1} \end{cases} \Rightarrow \begin{cases} c_1'(x) = x^{-1}\cos x \\ c_2'(x) = -x^{-1}\sin x \end{cases} \Rightarrow \begin{cases} c_1(x) = \int_\pi^x \frac{\cos t}{t}dt + C_1 \\ c_2(x) = -\int_\pi^x \frac{\sin t}{t}dt + C_2 \end{cases}$$

iv. 不显含未知量的方程：

$$F(x, y^{(n)}) = 0 \quad \Rightarrow \quad x = \varphi(t), \quad y^{(n)} = \psi(t), \quad y^{(n-1)} = \int \psi(t)\varphi'(t)dt + C_1, \ldots$$

v. 不显含自变量的方程：

$$F(y^{(n-1)}, y^{(n)}) = 0 \quad \Rightarrow \begin{cases} \Rightarrow y^{(n)} = f(y^{(n-1)}) \quad \Rightarrow z = y^{(n-1)}, z' = f(z) \Rightarrow \ldots \\ y^{(n-1)} = \varphi(t), \quad y^{(n)} = \psi(t), \quad x = \int \frac{\varphi'(t)}{\psi(t)}dt + C_1, \ldots \end{cases}$$

$$F(y^{(n-2)}, y^{(n)}) = 0 \quad \Rightarrow y^{(n)} = f(y^{(n-2)}) \quad \Rightarrow z = y^{(n-2)}, d(z'^2) = 2f(z)dz \Rightarrow \ldots$$

例如 P51：$\frac{d^2r}{dt^2} = f(r)$，不显含自变量的方程，可以在方程的两边同乘于 $2\frac{dr}{dt}$，则：

左边为 $2\frac{dr}{dt}\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right)^2$，右边为 $2\frac{dr}{dt}f(r) = 2\frac{dF(r)}{dt}$

因此 $\left(\frac{dr}{dt}\right)^2 = 2F(r) + C_1$，现在是一个可分离变量方程，易得结果。

再如 P19：$\begin{cases} \frac{d^2V(z)}{dz^2} = 4\pi G\rho_0 e^{-\beta V(z)} \\ V(0) = V'(0) = 0 \end{cases} \Rightarrow \left(\frac{dV(z)}{dz}\right)^2 = \frac{8\pi G\rho_0}{\beta}(1 - e^{-\beta V(z)})$

因为 $\rho(z) = \rho_0 e^{-\beta V(z)}$ 即 $V(z) = -\frac{1}{\beta}\ln\frac{\rho}{\rho_0}$，所以 $\left(\frac{d\rho(z)}{dz}\right)^2 = 8\pi G\beta(\rho_0 - \rho)\rho^2$

$$\left(\frac{d\bar{\rho}}{d\bar{z}}\right)^2 = 4(1-\bar{\rho})\bar{\rho}^2, \quad \bar{\rho} = \rho/\rho_0, \quad \bar{z} = z\sqrt{2\pi G\rho_0\beta}$$

$$\frac{d\bar{\rho}}{d\bar{z}} = -2\bar{\rho}\sqrt{1-\bar{\rho}}, \quad -\frac{d\bar{\rho}}{2\bar{\rho}\sqrt{1-\bar{\rho}}} = d\bar{z}, \quad \frac{1}{2}\ln\frac{1+\sqrt{1-\bar{\rho}}}{1-\sqrt{1-\bar{\rho}}} = \bar{z} + C, \quad C = 0$$

$$\operatorname{artanh}\sqrt{1-\bar{\rho}} = \bar{z}, \quad \bar{\rho} = (\operatorname{sech}\bar{z})^2 = (\cosh\bar{z})^{-2}$$

$$V(z) = -\frac{1}{\beta}\ln\bar{\rho} = -\frac{1}{\beta}\ln(\cosh\bar{z})^{-2} = \frac{2}{\beta}\ln\cosh\bar{z}$$

4. 微分方程组：$y_i' = f_i(x, y_1, y_2, ..., y_n), \quad i = 1, 2, ..., n$

线性齐次方程组：$y_i' = \sum_{j=1}^{n} a_{ij} y_j, \quad i = 1, 2, ..., n$

令 $y_i = A_i e^{kx}$，则 $kA_i = \sum_{j=1}^{n} a_{ij} A_j$ 即 $\sum_{j=1}^{n}\left(a_{ij} - k\delta_{ij}\right)A_j = 0$ 则 $\left|a_{ij} - k\delta_{ij}\right| = 0$，特征方程

通解：$y_i = \sum_{j=1}^{n} c_j A_j e^{k_j x}$

5. 偏微分方程：$F(x_1, x_2, ..., x_n, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, ..., \frac{\partial^m y}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}}), \quad m = m_1 + m_2 + \cdots + m_n$

分离变量：例如齐次波动方程：$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

令 $u(x,t) = X(x)T(t)$，则 $X(x)T''(t) = a^2 X''(x)T(t)$ 即 $\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} \triangleq -\lambda$

$\begin{cases} T''(t) + \lambda a^2 T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$，其中 λ 受到边值条件的限制

积分变换：例如轴对称调和方程：$\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0$

令 $\overline{u}(\xi, z) = \int_0^\infty r J_0(\xi r) u(r,z) dr$，因为 $\int_0^\infty r J_0(\xi r) \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) dr = -\xi^2 \overline{u}(\xi, z)$

所以 $\frac{\partial^2 \overline{u}(\xi, z)}{\partial z^2} - \xi^2 \overline{u}(\xi, z) = 0 \Rightarrow \overline{u}(\xi, z) = A(\xi) e^{\xi z} + B(\xi) e^{-\xi z}$

则 $u(r,z) = \int_0^\infty \xi J_0(\xi r) \overline{u}(\xi, z) d\xi$

6. 积分变换及其反演：

$$I_f(s) = \int_0^\infty f(t) K(s,t) dt; \quad f(t) = \int_0^\infty I_f(s) H(s,t) ds$$

若 $H(s,t) = K(s,t)$，则称 $K(s,t)$ 为 Fourier 核。

A. Fourier 余弦变换：

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\alpha x) dx; \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos(\alpha x) d\alpha$$

B. Fourier 正弦变换：

$$F_s(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha x)dx;\quad f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\alpha)\sin(\alpha x)d\alpha$$

C. Fourier 变换：

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{i\alpha x}dx;\quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}d\alpha$$

$$F(\xi,\eta)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y)e^{i(\xi x+\eta y)}dxdy;\quad f(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(\xi,\eta)e^{-i(\xi x+\eta y)}d\xi d\eta$$

D. ν 阶 Hankel 变换：

$$H(s)=\int_0^\infty tf(t)J_\nu(st)dt;\quad f(t)=\int_0^\infty sH(s)J_\nu(st)ds$$

E. Laplace 变换及其反演：

$$F(s)=\int_0^\infty f(t)e^{-st}dt;\quad f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)e^{st}dt$$

F. 梅林变换：

$$M(s)=\int_0^\infty f(t)t^{s-1}dt;\quad f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}M(s)t^{-s}ds$$

G. 有限 Fourier 余弦变换：

$$\bar{f}_c(n)=\int_0^\pi f(x)\cos(nx)dx;\quad f(x)=\frac{1}{\pi}\bar{f}_c(0)+\frac{2}{\pi}\sum_{n=1}^\infty \bar{f}_c(n)\cos(nx)$$

H. 有限 Fourier 正弦变换：

$$\bar{f}_s(n)=\int_0^\pi f(x)\sin(nx)dx;\quad f(x)=\frac{1}{\pi}\bar{f}_s(0)+\frac{2}{\pi}\sum_{n=1}^\infty \bar{f}_s(n)\sin(nx)$$

7. 积分：

A. 定积分：Riemann 和的极限：$\int_a^b f(x)dx = \lim_{\max\{\Delta x_i\}\to 0}\sum_i f(\xi_i)\Delta x_i$

B. Stieltjes 积分(P201): $g(x)$ 在区间[a,b]上无奇异点，则：

$$\int_a^b f(x)dg(x) = \lim_{\max\{\Delta x_i\}\to 0}\sum_i f(\xi_i)\big(g(x_{i+1})-g(x_i)\big)$$

C. 分部积分：$\int udv=u\cdot v-\int vdu \qquad \because (u\cdot v)'=u'\cdot v+u\cdot v'$

D. Dirichlet 积分：$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$

E. Fresnel 积分：$\int_0^\infty \sin x^2dx=\int_0^\infty \cos x^2dx=\frac{1}{4}\sqrt{2\pi}$

8. 数值方法：

A. Newton 迭代：

i. 非线性方程：$f(x)=0$

ii. 迭代格式：$x^{k+1}=x^k-\dfrac{f(x^k)}{f'(x^k)}$

iii. 思想源于 Taylor 展开 $0\equiv f(x)\approx f(x_0)+f'(x_0)(x-x_0)$

B. Newton- Raphson 迭代：

i. 非线性方程组：$\Phi_i(x_1,x_2,...,x_N)=0,\qquad i=1,2,...,N$

ii. 迭代格式：$x_i^{k+1}=x_i^k-\left[J_{ij}^k\right]^{-1}\Phi_j^{\,k}$，$J_{ij}=\partial\Phi_i/\partial x_j$

C. 最小二乘法：(习题 P182.6)

i. 问题提法：已知数据点(x_i,y_i)，$i=1,...,N$，线性拟合：$y=a+bx$

ii. 矛盾方程组：$\begin{cases} y_1=a+bx_1 \\ \ldots \\ y_N=a+bx_N \end{cases} \Leftrightarrow \begin{bmatrix} 1 & x_1 \\ \ldots & \ldots \\ 1 & x_N \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{Bmatrix} y_1 \\ \ldots \\ y_N \end{Bmatrix}$

iii. $$\begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_N \end{bmatrix}\begin{bmatrix} 1 & x_1 \\ \ldots & \ldots \\ 1 & x_N \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_N \end{bmatrix}\begin{Bmatrix} y_1 \\ \ldots \\ y_N \end{Bmatrix}$$

iv. $$\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^{\,2} \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{Bmatrix} \sum y_i \\ \sum x_i y_i \end{Bmatrix} \text{ i.e. } \begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^{\,2} \end{bmatrix}^{-1}\begin{Bmatrix} \sum y_i \\ \sum x_i y_i \end{Bmatrix}$$

拟合 $y=ae^{bx}$，改写 $\ln y=\ln a+bx$

思考：$y=a+bx+cx^2$