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# Exploring peculiarities of traffic flows with a viscoelastic model

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#### Exploring peculiarities of traffic flows with a viscoelastic model

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In this paper, a visco-elastic traffic flow model is described and applied to explore peculiarities of traffic flows on a loop road with ramp effects numerically. Based on different expressions for traffic pressure, sound speed and relaxation time, the viscoelastic model is derived from mass and momentum conservations and a linear viscoelastic constitutive relation. Numerical simulations of loop traffic flows have been carried out, with the ramp flow rate being assumed to be randomly dependent on the local main flow at the intersection. The results revealed that the on-ramp effect can cause the occurrence of traffic shock waves, the off-ramp effect can lead to the decrease of traffic density on the loop road. The viscoelastic effect does cause the significant changes of traffic flow pattern, indicating that self-organisation of loop traffic is a crucial impacting feature.

Keywords: viscoelastic traffic model; ramp flow effect; relaxation time; elasticity

#### 1. Introduction

Traffic flow is a hot topic in the societies of transportation research as well as physical-mathematics, because traffic flow has a crucial impact on people's travel trip planning and selection. As a result, a number of traffic flow models have been developed. The well-known one is the LWR model (Lighthill and Whitham 1955; Richards 1956) which is based on vehicular mass conservation, and probably the simplest for capturing the crucial features of traffic flows on highways, in contrast to a predicted steeper shock wave front (Kuhne and Michalopoulos 2001). The LWR model has been used to analyse interrupted traffic flow (Michalopoulos et al. 1984), to develop macroscopic models and compare with real data in Paris (Papageorgiou and Blosseville 1989), and to construct the entropy solutions with a discontinuous fundamental diagram (Lu et al. 2009). The LWR model extensions can certainly predict traffic hysteresis (Wong and Wong 2002), simulate the evolution of density waves (Zhu and Wu 2003) and the critical transition in bottleneck-related traffic (Chang and Zhu 2006), even though to a great extent the LWR-type model cannot predict highway traffic flows satisfactorily.

The fundamentals of transportation and traffic operations have been described in the work of Daganzo (1997). If the vehicular momentum conservation is considered as a fundamental feature in mathematical modelling, the proposed traffic flow models are in the category of high-order models, among which we should mention the historical work of Payne (1971) and the gas-kinetic analogy of Helbing and Treiber (1998). High-order models are useful in

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elucidating the stop and moving waves in traffic. For these models, despite of the unfavourable comment (Daganzo 1995), there are remarkable applications (Liu and Lyrintzis 1996; Ou and Dai 2006; Zhang and Wong 2006) and the further developments (Aw and Rascle 2000; Klar and Wegener 2000; Kiselev et al. 2000, 2004; Zhang 2003; Zhang and Wong 2006; Lebacque and Mammar 2007; Li 2008; Mammar, Lebacque, and Salem 2009; Zhang, Wong, and Dai 2009; Ngoduy 2013; Tordeux et al. 2014).

By introducing an average length of vehicles and a braking distance of traffic flows at free speed, Kiselev et al. (2000) have derived an expression of traffic sound speed and found the analytically obtained value for a VAZ-type vehicle agree well with the sound speed experimentally measured in Lincoln Tunnel in New York. They have further derived the expressions for propagation speed of traffic shock waves moving backward or forward.

By using the weak solution theory and the Payne–Whitham (PW) model (Payne 1971; Whitham 1974) to solve the travelling wave solution of a wide cluster, Zhang and Wong (2006) have shown the essence of conservation forms, and revealed that the conservation form for the acceleration equation is an important ingredient in the development of higher order traffic flow models. Furthermore, to describe traffic dynamics, Zhang, Wong, and Dai (2009) have considered three regimes which include the introduction of a pseudo-density transformed from the velocity, the pressure as a function of the pseudo-density and the relaxation of velocity to equilibrium. The resultant characteristic variables can be used to measure the deviation of the phase state to a desired state and derive physically bounded solutions. For instance, taking the pseudo-density as a conserved variable, the approach can be applied to describe both equilibrium and non-equilibrium flows in a systematic and unified manner, and thus complex traffic phenomena. Employing the traffic speed–density relation reported by Castillo and Benitez (1995a, 1995b), they modelled anisotropic traffic flow at three proposed conditions, and presented numerical examples to show the ability of the model to reproduce some notable traffic phenomena.

For travelling wave solutions of a quasi-linear hyperbolic system which can be described by the PW model, a stability analysis by virtue of a weighted energy method (Li 2008) has shown that the travelling wave solutions are asymptotically stable under small disturbances and under the subcharacteristic condition. The delicate balance between the relaxation and the diffusion that leads to the stability of the travelling waves is identified; namely, the diffusion coefficient is bounded by a constant multiple of the relaxation time. Using a gas-kinetic approach (Helbing and Treiber 1998; Hoogendoorn and Bovy 2000; Ngoduy 2012), a macroscopic traffic flow model has been developed (Ngoduy 2013), in the traffic flow intelligent vehicles are moving closer to each other than manual vehicles and operating in a form of many platoons each of which contains several vehicles. The model gives well tractable macroscopic equations for such platoon-based traffic operation. A linear stability analysis found that platoon-based driving behaviour of intelligent vehicles enhances the stabilisation of traffic flow with respect to a small perturbation. Numerical simulation of an open freeway with an on-ramp bottleneck supports the analytical results.

Tordeux et al. (2014) have reported the main aspects of a stochastic conservative model of the evolution of the number of vehicles per road section. The model defined in continuous time on a discrete space, follows a misanthrope Markovian process. It is a mesoscopic traffic model in the sense: the vehicles are individually considered, but their dynamics are aggregated per section. Supply and demand functions in equilibrium are taken as the model parameters.

Some problems involving traffic flows have been explained by applying and extending methods in statistical physics and nonlinear dynamics to self-driven many particle systems (Helbing 2001). A phase diagram, which was obtained for the nonlocal, gas-kinetic traffic model in the study of a freeway with ramp in the presence of single perturbation, was presented and explained (Helbing, Hennecke, and Treiber 1999).

(1)

In this paper, we use a fluid dynamic-type viscoelastic traffic flow model (Smirnova et al. 2014) to explore the flow peculiarities on a loop road when the on- and off-ramp effect exists. The traffic viscosity and elasticity are determined by traffic sound speed and relaxation time. The sound speed is represented approximately by the traffic operation parameters: the free flow speed, the jam density, and the flow transitional density if the jam pressure in traffic flows is identical to the total pressure at the flow transitional point. The relaxation time is supposed to be the travel time of an infinitesimal disturbance through a given distance. Based on the model and the total variation diminishing scheme (Roe 1981), numerical test is encompassed to obtain traffic flow patterns illustrated by density or speed contours in the temporal and spatial plane. These patterns are found to be affected by the traffic viscosity and elasticity relating the drivers' self-organisation, indicating that the viscoelastic model can provide some fluid dynamical evidences of why the traffic systems have a rich spectrum of pattern formation phenomena.

#### 2. Viscoelastic model

In this section, we briefly introduce the viscoelastic model. We see that traffic flow has its own self-organising behaviour, because vehicle drivers can usually adjust the moving speed so that the time headway can approach  $1/q_e$  when the instantaneous traffic flow rate (q) is unequal to the equilibrium one ( $q_e$ ) (Castillo 2001). This self-organisation has been considered by using the traffic viscosity and pressure in some high-order models (Kerner and Konhäuser 1993; Hilliges and Weidlich 1995).

The interaction of vehicles in congested flow is relatively extensive, causing the generation of a synchronised flow mode in which the flow rate is oscillating, or a jam exists (Kerner and Konhäuser 1993). However, the flow is homogeneous and stable for denser traffic when the traffic speed is lower than second critical speed corresponding to  $\rho_{c2}$  (Schönhof and Helbing 2009) Figure 1.

Remembering that the relaxation time is used for the traffic external force description, while the relaxation and elastic processes are intrinsically related according to the fundamentals of fluid mechanics, it is therefore relatively reasonable to further include the elastic effect in traffic mathematical modelling. In the model to be reported below, only the interaction between the lead and follower vehicles is considered, with the lane interaction being neglected. As a result, both the traffic flow rate and density can be considered as one-dimensional variables.

For a linear viscoelastic fluid flow, the shear stress can be expressed as



Figure 1. Fundamental diagram used by Kiselev et al. (2000).

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Figure 2. Schematic of the loop traffic flow with ramps.

here f(s) is the memory function. Based on the experimental observation of the relaxation of shear stress of macromolecule polymer and the theory of micro-rheology (Wagner 1978), we can write the memory function f(s) in the following form:

$$f(s) = G \sum_{1}^{N} \frac{1}{\tau_j} \exp\left(\frac{-s}{\tau_j}\right).$$
<sup>(2)</sup>

Here G is the modulus of fluid elasticity, and  $\tau_i$  is the relaxation time with the *j*th order.

For simplicity, we assume: (i) The road capacity is insensitive to the vehicle drivers; (ii) The traffic flow is one-dimensional, and satisfies the linear viscoelastic constitutive relation; (iii) The ramp effect is permitted, as schematically shown in Figure 2. Relating the operational conditions of main road and ramps, to accurately describe the on- and off-ramp flows themselves instantaneously is rather difficult. Hence, corresponding assumptions should be inevitable.

The assumption (ii) is based on the primary reason: we see that relaxation time in fact has been used in most of the high-order models to describe the driven force of vehicles, the relaxation time itself is a concept involving the elastic and viscous properties of fluids (Han 2000). Drivers' concerns of driving safety lead to the motion of vehicles has a memory behaviour. Therefore, it is appropriate to describe this traffic performance with a memory function.

The model derivation is initiated from the linear viscoelastic constitutive equation

$$\mathbf{T} = -p\mathbf{I} + G\sum_{j=1}^{N} \int_{0}^{\infty} \frac{1}{\tau_{j}} \exp\left(\frac{-s}{\tau_{j}}\right) \mathbf{H}(s) \,\mathrm{d}s.$$
(3)

where  $\mathbf{T}(=-p\mathbf{I} + \mathbf{T}_s)$  and p are, respectively, the stress tensor and the traffic pressure.  $\mathbf{H}(s)$  is the Finger deformation tensor. For the maximum relaxation order denoted by N, the Finger deformation tensor is given by  $\mathbf{H}(s) = \sum_{k=1}^{N} (-1)^{k+1} \frac{s^k}{k!} \mathbf{B}_k$ , where  $\mathbf{B}_k$  is the White–Metzner tensor, and s is the elapsed time period (Han 2000).

In the analogy to the unsteady traffic flows, using the first-order approximation in the case of N = 1, and the integration formula  $\int_0^\infty s^k \exp(-as) ds = k!/a^{k+1}$  merely valid for the positive integer k and the positive real number a, the traffic flow stress can be expressed as:

$$T = -p + G\tau B_1. \tag{4}$$

If the velocity of the traffic flow is u, then, according to Appendix 1,  $B_1 = 2u_x$  and then  $\rho v = 2G\tau$ ,  $T = -p + \rho v u_x$ . It is noted that for N = 2, a mathematical modelling and numerical evaluation are reported by Zhu and Yang (2013). They found the numerically predicted second

critical speed of traffic flow is consistent with the critical speed obtained from the theoretical analysis, if the flow–density relation is in a triangular form.

According to the existing high-order models, the general form of the forces acting on vehicular clusters can be written as

$$F = \frac{(q_e - q)}{\tau} + T_x.$$
(5)

where  $q_e$  is the traffic flow rate under the equilibrium traffic state, it can be seen as a monotonic function of traffic density, the subscribe 'e' represents the relevant variables taken under the equilibrium traffic state.  $T_x$  is the relevant surface force related to the traffic stress.

For simplicity, with the explanation given in the forgoing section, we employ the traffic fundamental diagram in Kiselev et al. (2000), which has the form of

$$q_e = \begin{cases} v_f \rho & \text{for } \rho \le \rho_*, \\ -c_\tau \rho \ln\left(\frac{\rho}{\rho_m}\right) & \text{for } \rho_* < \rho \le \rho_m, \end{cases}$$
(6)

As seen in Appendix 2, the traffic pressure can be written as

$$p = \frac{p_m (1 - \alpha)(\rho/\rho_m)}{[1 - \alpha(\rho/\rho_m)]},$$
(7)

where the maximum permissible density at free flow speed  $v_f$  is

$$\rho_* = \rho_m \exp\left(\frac{-v_f}{c_\tau}\right),$$

this definition can be derived directly the traffic state equation (6) by setting  $\rho$  approach  $\rho$ \*. Because the safe traffic density is taken to mean that for which the distance between vehicles are no shorter than the braking distance  $X(v_f)$ , the density  $\rho_*$  should be defined by the equality

$$\rho_* = \rho_m \left[ 1 + \frac{X(v_f)}{l} \right]^{-1}$$

Hence, from the two expressions of  $\rho_*$  given above, we have

$$c_{\tau} = \frac{v_f}{\ln[1 + X(v_f)/l]}$$

The traffic state equation (6) was shown by the fundamental diagram in Figure 1, where  $e \approx 2.71828$ ,  $\rho_{c2}$  is the second critical traffic density beyond which the traffic flow becomes stable again at a congested condition.

Note that  $\alpha = l\rho_m$ , *l* is the average length of vehicles,  $X(v_f)$  is the free flow speed-dependent braking distance,  $\rho_m$  is the jam density,  $p_m$  is the jam pressure,  $v_f$  is the free flow speed. Note that the flow–density fundamental diagram, usually called traffic state equation, apparently has a crucial impact on the traffic road operation (Haight 1963). For the urban traffic, a single parameter state equation has been reported and discussed previously (Zhu et al. 2002).

It should be remindful that traffic pressure – is just a notation, it does not exist physically. There is no actual inter vehicular pressure or momentum flux. The pressure occurs in crush tests, or when vehicles are colliding. In traffic flows there exist some rules of conduct, which regulate vehicles acceleration/deceleration via drivers' reaction to road conditions. Those rules of conduct could be determined in different ways, but usually they are incorporated into the equations

in terms of local flow acceleration dependence on flow conditions being multiplied by traffic density the equations formally look like momentum transfer equations (though no momentum is transferred between vehicles, which do not touch each other) (Kiselev et al. 2000; Smirnov et al. 2000).

Therefore, virtual attraction and repulsion forces are introduced, which for one-dimensional flow could be represented as a derivative of some function. That function by fluid flow analogy is named traffic pressure. This viewpoint has been appropriately used in mathematical modelling of traffic flows, recently reported elsewhere (Smirnova et al. 2014).

Also, it should be remindful that the expression of pressure is obtained by supposing the pressure is proportional to the reciprocal of spatial headway of vehicles rather than the traffic density directly. The spatial headway is of course vehicle-length dependent. To get an expression of the jam pressure, it is further supposed the jam pressure is approximately identical to the total pressure in the transition point  $\rho_*$ . Hence, we have

$$p_m = \frac{(1 - \alpha \rho_* / \rho_m) \rho_* / \rho_m}{2(1 - \rho_* / \rho_m)} v_f^2 \cdot \rho_m = c_0^2 \rho_m,$$
(8)

where

$$c_0^2 = \frac{(1 - \alpha \rho_* / \rho_m) \rho_* / \rho_m}{2(1 - \rho_* / \rho_m)} v_f^2.$$
(9)

As shown in Appendix 2, using the sound speed definition in fluid dynamics, we can derive the sound speed in traffic flows

$$c = \frac{c_0 \sqrt{1 - \alpha}}{(1 - \alpha \rho / \rho_m)} \tag{10}$$

We should mention that for the pressure-spatial headway dependency, there merely exists a reasonability in theoretical aspect. While justifications in traffic reality require to be further sought and explored. For instance, as soon as the spatial headway tends to zero, the traffic pressure must approach to infinity, implying the traffic speed should become zero abruptly. The assumption of jam pressure is made just for the sound speed derivation, since we are describing a fluid dynamic-type viscoelastic model.

Therefore, the governing equations of the viscoelastic traffic flows can be written as

1

$$\rho_t + q_x = m,$$

$$\rho(u_t + uu_x) = R,$$
(11)

where R satisfies the following equation:

$$R = \frac{\rho(u_e - u)}{\tau} - c^2 \rho_x + [(2G\tau)u_x]_x.$$
 (12)

We assume that the product of relaxation time and sound speed is a constant, indicating that there is a density dependence expressed as

$$\tau = \frac{\tau_0 (1 - \alpha \rho / \rho_m)}{\sqrt{1 - \alpha}} \tag{13}$$

where  $\tau_0 = l_0/c_\tau$ ,  $l_0$  is a characteristic length scale in traffic flows. Clearly, the relaxation time decreases linearly with density in traffic flow, which should be consistent with the general view of vehicular passengers.

The primary difference of the present traffic model from the visco-elastic model reported elsewhere (Zhu and Yang 2013), is that the model uses the first-order analogy approximation, the non-triangular traffic flow-density relationship, such as used by Kiselev et al. (2000), and includes the traffic ramp effect. In the present model, the traffic pressure, viscosity, relaxation time, and sound speed have been described individually.

#### 3. Numerical method

Choosing  $\rho$  and q as the mandatory variables, and  $R_1(=R + c^2\rho_x + \dot{m}u)$  instead of R, the governing equations of viscoelastic traffic flows have the form of

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S},\tag{14}$$

where  $\mathbf{U} = (\rho, q)^{\mathrm{T}}$ ,  $\mathbf{F}(\mathbf{U}) = (q, q^2/\rho + p)^{\mathrm{T}}$ , and  $\mathbf{S} = (\dot{m}, R_1)^{\mathrm{T}}$ , with the superscript 'T' representing a vector transpose.

The form of Equation (14) indicates that the two eigenvalues of the problem should be  $\lambda_k$ , (k = 1, 2), with  $\lambda_1 = u - c$  and  $\lambda_2 = u + c$ , since the Jacobian matrix is given by

$$\mathbf{A} = \begin{pmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + c^2 & 2u \end{pmatrix}.$$
 (15)

Note that in Transportation Society, there is a controversy for the eigenvalues. Some researches claim that the eigenvalue should not exceed the traffic speed, while others have no such a principle of traffic flow modelling. Actually, the present viscoelastic model uses the fluid dynamic analogy but not the car-following, the traffic speed is involving a vehicular cluster rather than a single vehicle. Hence, our viewpoint is that the eigenvalue limit should be abolishable.

The total variation diminishing (TVD) scheme (Roe 1981) is used to seek the numerical solutions of Equation (14). If we denote the space grid as  $x_i$ , and the time level as  $t^n$ , let the ratio of time step to grid step be  $\omega = \Delta t / \Delta x$ , the numerical stability condition of TVD is

$$\omega \cdot \max |\lambda_{k,i+1/2}| < 1, \quad k = 1, 2; \quad i = 0, 1, 2, \dots, N-1,$$
(16)

with  $\lambda_{k,i+1/2}$  representing the *k*th eigenvalue for **A** at  $x_{i+1/2}$  and *N* denoting the maximum of space grid number.

The term  $R_1$  is calculated at the time level  $t^n$  with a linear expansion of

$$R_1^{n+1/2} = R_1^n + \frac{1}{2} \left(\frac{\partial R_1}{\partial \rho}\right) \delta \rho^n + \frac{1}{2} \left(\frac{\partial R_1}{\partial q}\right) \delta q^n, \tag{17}$$

where  $\delta \rho^n = \rho^{n+1} - \rho^n$ ,  $\delta q^n = q^{n+1} - q^n$ . Let  $v_0$  and  $\Delta x (= l_0)$  denote the speed and length scales, respectively, we have

$$\frac{\partial R_1}{\partial \rho} = \tau^{-1} \left( \frac{\partial q_e}{\partial \rho} \right) + \sigma_1 u^2, \quad \frac{\partial R_1}{\partial q} = -\tau^{-1} + 2\sigma_1 u, \tag{18}$$

where  $\sigma_1 (= \dot{m}/q)$  is zero, at those sections denoted by  $x_i$  without ramp connection, while it has a assumed random number at ramp intersections. Note that the assumption for  $\sigma_1$  is closer to reality

Case	$\bar{\sigma}_1(x_{p1})$	$\bar{\sigma}_1(x_{p2})$	$\hat{G}_{\tau 0} = \left[ rac{2G( au_0  u_0)}{l_0^2} \cdot rac{t_0}{q_0}  ight]^{\mathrm{a}}$
I	0	0	0.03175
II	$-\frac{1}{3}$	$\frac{1}{8}$	0.03175
III	$-\frac{1}{3}$	$\frac{1}{6}$	0.03175
IV	$-\frac{1}{3}$	$\frac{1}{4}$	0.03175
V	0	0	0.0625
VI	0	0	0.08
VII	0	0	0.125

Table 1. The parameters for the simulation of loop traffic flows.

<sup>a</sup>Respectively, the flow rate, speed and time scales are  $q_0 = \rho_* v_f$ ,  $v_0 = l_0/t_0$ ,  $t_0 = l_0 \rho_m/q_0$ .



Figure 3.  $\sigma_1$  and  $\sigma_1 q$  plotted respectively as a function of time. (a)  $\sigma_1$ ; (b)  $\sigma_1 q$ .

than assigning a definite value to  $\dot{m}$ . The random number should be positive for a on-ramp, but negative for a off-ramp, and it is supposed to be assigned with the Gaussian normal distribution.

For a special case IV(see in Table 1), the evolutions of  $\sigma_1$  can be seen in Figure 3(a), the instant data of  $\sigma_1$  are recorded during the traffic flow prediction, which supposes that the ramp flow is randomly relates to the local instantaneous main road flow at the ramp intersections. In Figure 3(a), for the black curve, the mean value  $\bar{\sigma}_1$  for on-ramp at x = 220 (in the unit of  $l_0$ ) is  $\frac{1}{4}$ , but for the red curve at x = 30 for off-ramp, it is set as  $-\frac{1}{3}$ . The ratio of root-mean-square value  $\sigma'_1$  to its mean  $\bar{\sigma}_1$  is set as 0.03737, unless further note is given.

The TVD scheme can be written as

$$\delta \mathbf{U}_{i}^{n} = -\omega(\hat{\mathbf{F}}_{i+1/2} - \hat{\mathbf{F}}_{i+1/2}) + (\Delta t)\mathbf{S}_{i}^{n} + \frac{\Delta t}{2} \left(\frac{\partial \mathbf{S}}{\partial \mathbf{U}}\right)_{i}^{n} \delta \mathbf{U}_{i}^{n},$$
(19)

where  $\Delta t (= t^{n+1} - t^n)$ . The numerical flux  $\hat{\mathbf{F}}_{i+1/2}$  can be calculated by using the left and right eigen vectors of Jacobian matrix *A*. The calculation of  $\hat{\mathbf{F}}_{i+1/2}$  involves with the coefficient of viscous term  $Q_k(z)$  with a small artificial parameter  $\epsilon_k$ . Some details for the case of vanished source **S** can be found in the previous work reported elsewhere (Zhu and Wu 2003; Chang and Zhu 2006). it is noted that in the calculation of  $\Delta_k [= \omega(\hat{F}_{k,i+1/2} - \hat{F}_{k,i+1/2})$  for k = 1, 2],  $\rho_{i+1/2}$ and  $q_{i+1/2}$  are obtained by Shui (1998)

$$\rho_{i+1/2} = 0.25 (\rho_i^{1/2} + \rho_{i+1}^{1/2})^2,$$

$$q_{i+1/2} = \rho_{i+1/2} \cdot u_{i+1/2},$$
(20)

where  $u_{i+1/2} = (u_i \rho_i^{1/2} + u_{i+1} \rho_{i+1}^{1/2}) / (\rho_i^{1/2} + \rho_{i+1}^{1/2})$  is the weighted speed.

#### 4. Results and discussion

#### 4.1. Simulation parameters

To explore the viscoelastic effects of traffic flows, numerical simulations using the TVD (Roe 1981; Shui 1998) scheme were conducted for loop traffic with ramp effects. A periodic boundary condition is used, implying that solutions at  $x_N$  are superposed on those at  $x_0$ . The loop road length is assumed to be 250, has a length unit  $l_0 = 160$  m. The length unit ( $l_0$ ) determines the total loop road length when the total grid number is fixed. The velocity scale is chosen as  $v_0 = q_0/\rho_m (= v_f \rho_*/\rho_m \approx 3.176 \text{ m/s})$ , with the time scale  $t_0 = l_0/v_0 (\approx 50.377 \text{ s})$ . The off-ramp intersection is set at  $x_{p1} = 30$ , with the on-ramp intersection located at  $x_{p2} = 220$ .

The initial density conditions are given by

$$\rho(0,x) = \begin{cases} 0.9 & \text{for } x \in [124, \ 126], \\ 0.3 & \text{otherwise,} \end{cases}$$
(21)

with  $q(0,x) = q_e(\rho(0,x))$ , which can be calculated in terms of the fundamental diagram. The numerical simulations use the flow-density relation shown in Figure 1 and the traffic operation parameters given in Table 2.

When the average length of vehicles (*l*) is assumed to be 5.8 m, the traffic jam density is about 150 m. While when the free flow speed is fixed as 110 km/h, the braking distance can be assumed to be about 50 m, merely made on the basis of the discussion with some drivers having the driving experiences on real freeways. The braking distance depends on the vehicular moving speed, as seen in the expression for  $c_{\tau}$  occurred in the traffic state equation (6). With the values of  $X(v_f)$ , *l*, and  $v_f$ , the  $c_{\tau}$  value can be obtained. Correspondingly, when the traffic length scale  $l_0$ is fixed, the value of relaxation time  $\tau_0$  can be predicted, i.e.  $\tau_0 (= l_0/c_{\tau})$  is  $0.2353t_0 \approx 11.854$  s.

Table 2. The parameters of loop traffic operations.

v <sub>f</sub> (km/h)	$\rho_m$ (veh/km)	$X(v_f)$ (m)	<i>l</i> (m)
110	150	50	5.8

The viscoelastic parameter  $\hat{G}_{\tau 0}[= 2G(\tau_0 v_0)/l_0^2 \cdot t_0/q_0]$  is given in Table 1. The different value of  $\hat{G}_{\tau 0}$  for the case of V, VI, or VII is given by adjusting the modulus of fluid elasticity.

From Table 1, it is seen that seven cases were numerically studied. The off-ramp parameter  $\bar{\sigma}_1(x_{p1})$  is fixed to be  $-\frac{1}{3}$ , while the on-ramp parameter  $\bar{\sigma}_1(x_{p2})$  is assumed to be changeable. This assumption is helpful for numerically exploring the on-ramp effect on the loop traffic flows. Note that the ratio of root-mean-square value  $\sigma'_1$  to its mean  $\bar{\sigma}_1$  is set as 0.03737.

The ratio of time step to space grid step denoted by  $\omega$  is set with a Courant number of 0.75 (Shui 1998), i.e.  $\omega = 0.75/\max |\lambda_{k,i+1/2}|$ , for k = 1, 2, i = 0, 1, 2, ..., N - 1. The small artificial parameter  $\epsilon_k$  used in the coefficient of viscous term  $Q_k(z)$  for numerical fluxes in TVD is assumed to be 0.5% of the normalised free speed  $v_f/v_0$ .

Respecting the previous work of Zhu and Yang (2013), we choose the elasticity in a similar way. In the present work, we incline to employ the traffic fundamental diagram used previously (Kiselev et al. 2000) to explore the traffic flow characteristics on the basis of the viscoelastic model introduced in this paper.

#### 4.2. Viscoelastic effect

For the convenience of discussion, we present the viscoelastic effect when ramps vanishes, i.e. for the cases I, V, VI, and VII, as given in Table 1. For Case I, the traffic flow pattern is illustrated by the speed contours, as shown in Figure 4. There are at least six backward travelling traffic shock waves. These traffic shocks propagate reversely at speeds, which not only depend on the traffic operation parameters given in Table 2, but also intrinsically relate to the fundamental diagram (Daganzo 1997; Kiselev et al. 2000), because the present visco-elastic model has employed the fundamental diagram to determine the equilibrium traffic state, so that the external driving force of traffic flow [ $\rho(u_e - u)/\tau$ ], can be given explicitly.

It should mentioned that these shocks wave speeds have been expressed analytically at first in terms of local flow variables previously (Kiselev et al. 2000). Even without the ramp flow impacts, on the loop road, there exists interaction of the traffic shock and the deflation waves. Although the driving maneuver certainly has influences on traffic flow patterns, it is conditioned by the flow environment and traffic regulations. With the increase of time, the shock wave effect attenuates, implying the initially assigned heterogeneity can be gradually removed by the selforganisation of the loop traffic.

It is noted that in Figure 4, the red-coloured region has a speed beyond 0.6, in the unit of free flow speed  $v_f$ . The yellow-coloured region has a speed in the range of  $v \in (0.5, 0.6]$ . While in the green-coloured region, the traffic speed is in the range from 0.3 to 0.5, with the cyan-coloured region having speed in the range from 0.1 to 0.3. For instance, in the coarse irregular red lines



Figure 4. Traffic speed contours in the t - x plane for Case I. Note that the contours are labelled by values from 0.1, 0.3, 0.5, and 0.6 in the unit of  $v_f$ .



Figure 5. Traffic speed versus time at different sections. (a) x = 30, 220; (b) x = 125.

with positive slope, traffic flow has a speed larger than  $0.6v_f$ , even though the average density (0.3) on the loop road is much greater than the first critical value  $\rho_* \approx 0.104\rho_m$ . Obviously, the average density is chosen slightly less than the saturation density  $\rho_m/e$ , at which the equilibrium traffic flow rate arrives at its maximum value  $c_{\tau}/e$ . Here,  $e \approx 2.71828$ . It is seen that the mutually interaction of the traffic shock and the deflation waves has resulted in complicated traffic flow structures on the loop road.

Corresponding to the flow pattern given by Figure 4, the temporal evolution of traffic speed at three monitoring sections, i.e. x = 30, 125, and 220, are shown in Figures 5(a) and 5(b). It is seen that the speed curves oscillate like stop and moving waves (Castillo 2001; Kuhne and Michalopoulos 2001). Intrinsically, the speed oscillation depends on the interaction of shock and deflation waves. As a part of numerical solution, to a great extent, the speed evolution at a monitoring section is examined by the initial and boundary conditions.

The predicted speed and density at the three monitoring sections are plotted in Figures 6 and 7, for a comparison with the observation data, which are abstracted from McShane, Roess, and Prassas (1998) and labelled by non-filled squares. In Figure 6(a) and 6(b), the instantaneous traffic density ( $\rho$ ) and speed normalised by the free flow speed ( $u/v_f$ ), which were recorded at the two sections  $x_{p1}(=30)$  and  $x_{p2}(=220)$ , were illustrated by non-filled circles, together with the normalised equilibrium speed  $u_e/v_f$  shown by black triangles. While in Figure 7, the instantaneous density and speed were recorded at the section x = 125 and shown by non-filled circles and black triangles, respectively. It is seen that both figures show that the simulation results are reliable.

However, the traffic flow pattern varies with the increase in viscoelastic parameter  $\hat{G}_{\tau 0}$ , as seen in the loop traffic flow patterns shown in Figure 8(a) and 8(d). Since the parameter  $\hat{G}_{\tau 0}$  is naturally related to the viscosity and elasticity of traffic flows as indicated in Equation (4), and intrinsically involves with the driving maneuver of vehicles which is reflected by the



Figure 6. Comparison of traffic speed with measured data time for Case I at (a) x = 30, and (b) x = 220. Note that the record number is about 6700 in the 8.4 hours' simulation. The observation data are extracted from McShane, Roess, and Prassas (1998), and the jam density used in density normalisation is assumed to be 200 veh/mile.



Figure 7. Comparison of traffic speed with measured data time at x = 125. The observation data are extracted from Ref. McShane, Roess, and Prassas (1998) as given in the caption of Figure 6.

traffic self-organisation property, it can be therefore concluded that for the loop traffic flow without ramp effect, its self-organisation is a crucial feature in impacting on the spatial-temporal evolution of traffic waves.

#### 4.3. Ramp effect

Ramp effect is usually non-negligible. In fact, the interaction intensity between the ramp and loop traffic flows depends upon not only the on/off-ramp flow rates but also the loop traffic operational situation. Certainly, there are difficulties in artificially assigning the ramp flows that



Figure 8. Traffic speed contours in the t - x plane for (a) Case I, (b) Case V, (c) Case VI, and (d) Case VII. Note that the contours are labelled by values from 0.1, 0.3, 0.5 and 0.6 in the unit of  $v_f$ .

are well consistent with that occurred in reality, we use the mathematically simple approach based on random number setting, as described in Section 3.

We just present the numerical results for Cases I, II, III, and IV, with the viscoelastic parameter  $\hat{G}_{\tau 0}$  being assumed to be unchangeable, and equal to 0.03175. As shown by density contours in Figure 9(a)–(d), in the considered cases, the increase in  $\sigma_1(x_{p2})$  means a increase in on-ramp flow

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Figure 9. Traffic density contours in the t - x plane for (a) Case I, (b) Case II, (c) Case III, and (d) Case IV. Note that the contours are labelled by values from 0.1, 0.2, 0.3, and 0.5 in the unit of  $\rho_m$ .

rate, has resulted in the occurrence of traffic congested region labelled by red colour, indicating that due to the vehicle aggregation the density in the region is beyond  $0.5\rho_m$ . This *region* has a triangular shape, since there also has the off-ramp effect, the off-ramp section is set as  $x_{p1}$ . The off-ramp flow rate is assigned similar to that on-ramp, which is shown in Figure 3(b) for case IV.

The loop road density can be lower than  $0.1\rho_m$ , as seen in the blue colour regions in Figures 9(b)–(c). This indicates that a large amount of loop vehicles has been released from

the off-ramp. Some evidence can also be found in Table 1, in Cases II and III  $\bar{\sigma}_1(x_{p1}) = -\frac{1}{3}$  which was used to described the amount of mean off-ramp flow, its absolute value is larger than  $\bar{\sigma}(x_{p2}) = \frac{1}{8}, \frac{1}{6}$ , or  $\frac{1}{4}$  which was used for the amount of mean on-ramp flow. The traffic density can also be larger than  $0.5\rho_m$ , as seen in the red colour regions in Figure 9(a)–(d).

Comparing the flow patterns shown by density contours in Figure 9(a)-(d), it is seen that without the ramp effect as seen in Figure 9(a), due to self-organisation property, the loop traffic flow can attenuates the initial density heterogeneity. While with ramp effect, as shown by Figure 9(b)-(d) for the three cases II, III, and IV, the spatial-temporal evolution of traffic flows can be apparently different. What simulated are several simple loop traffic flows. In fact, for traffic flows in reality, there exists a surprisingly rich pattern formation reasons which are naturally existed in traffic flows, as reported elsewhere (Herman and Ardekani 1984; Nagatani 2002; Helbing, Hennecke, and Treiber 1999).

#### 5. Conclusions

Considering the relaxation time has been used to represent the external traffic force for a long time in continuum traffic flow modelling, it indicates that traffic flow has elasticity. A fluid dynamic-type viscoelastic traffic flow model is developed, in which the expressions for traffic pressure, sound speed, and relaxation time, have been derived with the basic parameters of traffic systems. Using this model, we have explored the traffic peculiarities of loop traffic flows with ramp effects.

Numerical tests reveal that for loop traffic flows, the viscoelastic effects can cause the variation of traffic flow pattern, the existence of complicated structures of shock and deflation waves due to mutually interaction, suggesting that the self-organisation property of vehicles is a crucial feature in traffic operations. For the cases of off-ramp flow rate is relatively large, the loop road density can be lower than its initial density assigned. The on-ramp flow is a reason causing the occurrence of traffic shock waves and vehicular aggregated region on the t - x plane. The present viscoelastic model has a potential in predicting traffic flows in reality.

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#### Appendix 1. Relation between $B_n$ and $B_{n+1}$

In Chapter II of Ref. Han (2000), it was written that for  $n \ge 1$ 

$$\mathbf{B}_{n+1}(s) = \frac{\mathrm{d}\mathbf{B}_n}{\mathrm{d}s} - \mathbf{L}_1 \mathbf{B}_n - \mathbf{B}_n \mathbf{L}_1^{\mathrm{T}},\tag{A1}$$

where  $\mathbf{L}_1 = \nabla \mathbf{u}, \mathbf{L}_1^{\mathrm{T}}$  is the transposition of  $\mathbf{L}_1$ .  $\mathbf{B}_0 = 1$ , and  $\mathbf{B}_1 = \mathbf{L}_1 + \mathbf{L}_1^{\mathrm{T}}$ .

#### **Appendix 2. Pressure derivation**

Remembering the assumption for traffic pressure, let the average length of vehicles be l, we have

$$p \propto \frac{1}{s-l},$$
 (A2)

where  $s = 1/\rho$ . Assume  $\alpha = l\rho_m$ , we can express traffic pressure as

$$p \propto \frac{\rho}{1 - \alpha \rho / \rho_m}.$$
 (A3)

Let the jam pressure be  $p_m$ , we have

$$p = \frac{p_m (1 - \alpha)(\rho/\rho_m)}{[1 - \alpha(\rho/\rho_m)]}.$$
 (A4)

Therefore, we can express the sound speed as

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{c_0^2 (1 - \alpha)}{(1 - \alpha \rho / \rho_m)^2},\tag{A5}$$

where, as given by Equation (8),

$$c_0^2 = \frac{(1 - \alpha \rho_* / \rho_m) \rho_* / \rho_m}{2(1 - \rho_* / \rho_m)} v_f^2.$$
 (A6)