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KINEMATIC WAVE PROPERTIES OF ANISOTROPIC DYNAMICS MODEL FOR TRAFFIC FLOW *

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Abstract: *The analyses of kinematic wave properties of a new dynamics model for traffic flow are carried out. The model does not exhibit the problem that one characteristic speed is always greater than macroscopic traffic speed, and therefore satisfies the requirement that traffic flow is anisotropic. Linear stability analysis shows that the model is stable under certain condition and the condition is obtained. The analyses also indicate that the model has a hierarchy of first- and second-order waves, and allows the existence of both smooth traveling wave and shock wave. However, the model has a distinctive criterion of shock wave compared with other dynamics models, and the distinction makes the model more realistic in dealing with some traffic problems such as wrong-way travel analysis.*

Key words: traffic flow; dynamics model; kinematic wave property

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Introduction

Recently the authors presented a new dynamics model for traffic flow[1]. The model comprises a continuum equation and a dynamics equation

$$\partial k/\partial t + \partial(ku)/\partial x = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{u_e(k) - u}{T} + a\frac{\partial u}{\partial x}, \tag{2}$$

where k is traffic density; u is mean speed; x, t are space and time coordinates respectively. T is relaxation time; a is propagation speed of disturbance; $u_e(k)$ is equilibrium speed density relationship. Eq.(1) indicates that the number of vehicles on the road is in conservation. The left-hand side of (2) is the acceleration of vehicles. The first term on the right-hand side of (2) is relaxation term, representing the process that driver adjusts the speed of the vehicle to equilibrium; the second term is anticipation term, representing the process that driver reacts to the traffic ahead. Compared with other correlative dynamics models, such as Payne model made up of the continuum Eq.(1) and the following dynamics equation[2]

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$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{u_e - u}{T} - \frac{a^2}{k}\frac{\partial k}{\partial x}, \tag{3}$$

they have the same continuum equation, but in the new model, speed gradient term replaces density gradient in the anticipation effect of the dynamics equation. Just this replacement enables the new model to remove the problem that one characteristic speed is always greater than macroscopic traffic speed. The two characteristic speeds of Eqs. (1) and (2) can be calculated as follows:

$$\lambda_1 = u - a, \quad \lambda_2 = u, \tag{4}$$

while for Payne model (1) and (3), the two characteristic speeds are

$$\lambda_1 = u - a, \quad \lambda_2 = u + a. \tag{5}$$

It is obvious that the second characteristic speed λ_2 in Payne model is greater than the macroscopic traffic speed u, but the new model does not exhibit this problem. Since the characteristic speed greater than u implies that vehicle will be affected by what is happening behind it, therefore the fundamental principle of traffic flow that vehicles are anisotropic and respond only to that frontal stimuli is violated[3]. From this point of view, the new model is more realistic.

The controlling Eqs. (1) and (2) of the new model constitute hyperbolic equations, and study of kinematic wave phenomena in hyperbolic equations is very important, thus it is necessary to carry out the analyses of the kinematic wave properties of the new model. In the following paragraphs we discuss the linear stability, wave hierarchy and shock condition respectively.

1 Linear Stability Analysis

Assuming k_0 and $u_0 = u_e(k_0)$ are the steady state solutions of Eqs. (1) and (2), $k = k_0 + \xi$ and $u = u_0 + \eta$ are perturbed solutions of (1) and (2), with $\xi = \xi(x, t)$ and $\eta = \eta(x, t)$ small perturbations to the steady state solutions. Next we discuss how these perturbations evolve over time. Substituting the perturbed solutions $k = k_0 + \xi$, $u = u_0 + \eta$ into (1) and (2), then taking Taylor series expansions of the perturbed equations at k_0 and u_0, and neglecting higher order terms of ξ and η, we obtain the following linearized equations

$$\xi_t + u_0\xi_x + k_0\eta_x = 0; \tag{6}$$

$$\eta_t + u_0\eta_x = \frac{u_e'(k_0)\xi - \eta}{T} + a\eta_x, \tag{7}$$

where $u_e' = \mathrm{d}u_e/\mathrm{d}k$. Making derivative to x on both sides of (7), we have

$$\eta_{tx} + u_0\eta_{xx} = \frac{u_e'(k_0)\xi_x - \eta_x}{T} + a\eta_{xx}. \tag{8}$$

From (6), we obtain the relation

$$\eta_x = -(\xi_t + u_0\xi_x)/k_0. \tag{9}$$

Making derivative to x, t on both sides of (9) respectively, we have $\eta_{xx} = -(\xi_{tx} + u_0\xi_{xx})/k_0$, $\eta_{tx} = -(\xi_{tt} + u_0\xi_{tx})/k_0$. Substituting η_{xx}, η_{tx}, η_x into (8), and assuming $c_0 = (ku)'\,|_{k=k_0} = u_0 + k_0u_e'(k_0)$, we can eliminate η, and obtain the following second-order equation

$$(\partial_t + c_0\partial_x)\xi = -T[(\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)\xi], \tag{10}$$

where

$$c_1 = u_0 - a, \; c_2 = u_0. \tag{11}$$

According to the traditional way of linear stability analysis, considering the perturbation has the exponential form $\xi(x, t) = \xi_0 \exp i(\gamma x - \omega t)$, substituting the form into (10), we have

$$(-i\omega + ic_0\gamma)\xi = -T[(-i\omega + c_1 i\gamma)(-i\omega + c_2 i\gamma)]\xi. \qquad (12)$$

For ξ to be non-trivial solution of (12), we must have

$$T(\omega - c_1\gamma)(\omega - c_2\gamma) + i(\omega - c_0\gamma) = 0. \qquad (13)$$

So the linear stability of the new model is determined by the imaginary parts of the solutions of the quadratic Eq. (13) about ω. The new model is unstable if the imaginary part of one of the solutions is positive; only when the imaginary parts of both solutions are less than or equal to zero can the new model be stable. For equation having the form as (13), Whitham[4] and Zhang[5] pointed out, when the following inequality is met, it can be guaranteed that the imaginary parts of both solutions are less than or equal to zero

$$c_1 \leqslant c_0 \leqslant c_2. \qquad (14)$$

This is the linear stable condition of the new model. We may explain the physical sense of the condition in next section.

2 Wave Hierarchy

In this section, we discuss wave hierarchy of the new model. Note that the right-hand side of (10) is a second-order wave operator, where c_1, c_2 are the characteristic speeds of the second-order wave. The left-hand side of (10) is a first-order wave operator, where c_0 is the characteristic speed of the first-order wave. Eq. (10) reveals that kinematic wave of the new model is composed of two hierarchies: first-order wave and second-order wave.

Whitham[4] gave the second-order linearized equation of Payne model, which has the same form as (10), the difference is that c_1 and c_2 have a different expressions. In Payne model

$$c_1 = u_0 - a, \quad c_2 = u_0 + a. \qquad (15)$$

Comparing (4) and (5) with (11) and (15), we find that the characteristic speeds of second-order wave is very similar to the characteristic speeds of the corresponding model. For linearization reason, u is replaced by the steady state speed u_0.

From above analyses, we learn that in both the new model and Payne model, the wave has two hierarchies, where first-order wave propagates with the same speed c_0, and the second-order wave propagates with different speeds which are determined by the characteristic speeds of the model.

It is known that the characteristic speeds c_1 and c_2 of second-order wave are slowest and fastest propagation speeds of the perturbation signals, and the characteristic speed c_0 of first-order wave is the propagation speed of the main perturbation signals. When $c_1 \leqslant c_0 \leqslant c_2$ is satisfied, there will be no conflict between the propagation of different waves, thus the system is stable. On the other hand, when $c_0 > c_2$ or $c_0 < c_1$, the propagation speed of main signals is greater than fastest propagation speed of signals or less than slowest propagation speed of signals, conflict between the propagation of different waves arises, which leads to an unresolvable competition between first-order wave and second-order wave. As a consequence of the competition, the system loses the stability. These explain the physical sense of the stable condition (14).

3 Shock Condition

Whitham[4] showed that in Payne model, two constant states $(k, u)_{1,2}$ can be connected by a smooth steady wave traveling at a constant speed U if the following condition is met

$$u - a < U < u + a. \tag{16}$$

When the condition is not satisfied, the two constant states may be connected by a shock. In this section, we will discuss whether it is the case in the new model.

Considering a steady profile solution with a constant translational speed U

$$k = k(X), \quad u = u(X), \quad X = x - Ut.$$

Substituting the steady profile solution into (1) and (2), we can obtain

$$-Uk_X + (uk)_X = 0, \tag{17}$$

$$T(u - a - U)u_X = u_e - u. \tag{18}$$

Integrating (17) and reformulating, we have

$$u = U - \frac{A}{k}, \tag{19}$$

where A is a constant. Substituting (19) into (18), and multiplying k on both sides, we obtain

$$-T(U - u)[U - (u - a)]k_X = q_e - Uk + A, \tag{20}$$

where $q_e = u_e k$. At constant states $(k, u)_{1,2}$, we have $k_X = 0$. So U and A should satisfy the following equation

$$q_e(k_1) - Uk_1 + A = q_e(k_2) - Uk_2 + A = 0, \tag{21}$$

thus

$$U = \frac{q_e(k_1) - q_e(k_2)}{k_1 - k_2}. \tag{22}$$

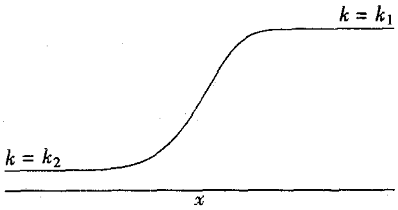

Fig.1 Smooth wave profile

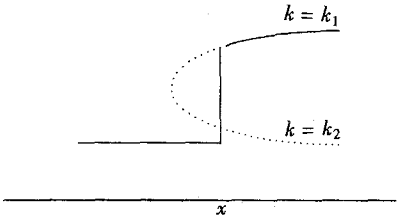

Fig.2 The profile that turns back and shock structure

Defining $h(k) = q_e(k) - Uk + A$. For traffic flow, $q_e(k)$ is strictly concave-down, i.e., $q_e''(k) < 0$, so

$$h''(k) = q_e''(k) < 0. \tag{23}$$

From (21) and (23), when $k \in (\min(k_1, k_2), \max(k_1, k_2))$, $h(k) > 0$. Furthermore, $q_e''(k) < 0$ also means that a steady wave connecting two constant states can only be compression wave[4,5], thus $k_1 > k_2$.* If $-T(U - u)[U - (u - a)]$ remains positive in the range

* Assuming k_1, k_2 represent densities at the location downstream (right)/upstream (left) of the steady profile respectively.

$k_2 < k < k_1$, then $k_X > 0$, we have a smooth profile as shown in Fig.1. Obviously solution to the inequality $-T(U-u)[U-(u-a)] > 0$ is

$$u - a < U < u. \tag{24}$$

Note that the left-hand side of (20) is proportional to relaxation time T. With the increase of T, $-T(U-u)[U-(u-a)]$ increases, and the profile becomes smoother and smoother.

When condition (24) is not satisfied, $-T(U-u)[U-(u-a)]$ changes sign in the profile, as shown in Fig.2, and a single-valued continuous profile is no longer possible, the profile turns back itself. When this is the case, the profile can be rectified by fitting in an appropriate shock as shown in Fig.2. *

Comparing (16) with (24), the difference between two models is that u in the new model replaces $u + a$ in Payne model. Next we examine the different results caused by this difference through an example. The initial/boundary conditions are as follows (as shown in Fig.3):

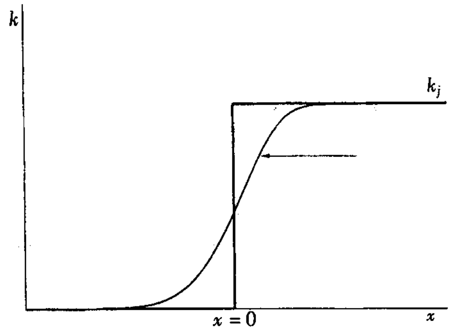

Fig.3 The incorrect smooth profile to the initial/boundary conditions (25), (26)

$$u = 0,\ k = k_j\mathrm{H}(x), \qquad \forall x \leqslant A,\ t = 0(A > 0), \tag{25}$$

$$u = 0, \quad x = A, \quad t > 0, \tag{26}$$

where $\mathrm{H}(x)$ is Heavyside step function, k_j is jam density.

Under these initial/boundary conditions, the correct solution is that there will be no movement of vehicles, i.e., there is a shock with zero speed. However, for Payne model, if a steady solution exists, then from (22) we have $U = 0$. Substituting $u = 0$ and $U = 0$ into (16), it is clear that (16) is met, i.e., the steady solution is a smooth profile, which implies wrong-way travel occurs (as shown in Fig.3). For the new model, substituting $u = 0$ and $U = 0$ into (24), (24) is not satisfied, thus it is a shock solution, no wrong-way travel arises. The analyses verify that characteristic speed $u + a$ does cause incorrect traffic flow, while that u in the new model replaces $u + a$ in Payne model can remove the problem.

4 Conclusions

In this paper, the kinematic wave properties of a new dynamics model for traffic flow recently presented by the authors have been studied, that includes linear stability, wave hierarchy and shock condition. The analyses show that the new model is stable under certain condition and we derive out the stable condition. It also indicates that kinematic wave of the new model has two hierarchies, i.e., first-order wave with characteristic speed c_0 and second-order wave with characteristic speeds c_1, c_2. For the reason that the first-order wave tries to violate the

* In traffic flow, apart from conservation of number of vehicles, it is not clear what conservation principle traffic speed obeys. Therefore it is somewhat difficult to obtain the correct shock speed. This difficulty is present in all dynamics model and is worthy of further investigation.

characteristic speeds of second-order wave, instability of traffic flow arises. Furthermore, the new model allows smooth profile solutions under some conditions, while under other conditions the profile that turns back itself should be rectified by inserting a proper shock.

Comparing with Payne model, we find out, although kinematic wave properties of the new model are similar to those of Payne model, important differences exist. The reason for the differences lies in that in the new model, characteristic speed greater than macroscopic traffic speed is removed. Just the differences enable the new model to describe traffic flow more realistically, which is verified by a traffic instance. From the analyses, it may be reasonable to conclude that the new model provides a more accurate and realistic description of traffic flow.

References:

[1] JIANG Rui, WU Qing-song, ZHU Zuo-jin. A new dynamics model of traffic flow[J]. *Chinese Science Bulletin*, 2001, **46**(4): 345 - 348.

[2] Payne H J. Models of freeway traffic and control[A]. In: G A Behey Ed. *Proc Simulation Council Mathematical Models of Public Systems*[C]. La Jolla: Simulation Council, 1971, **1**(1): 51 - 61.

[3] Daganzo C F. Requiem for second-order fluid approximations of traffic flow[J]. *Transp Res B*, 1995, **29**(4): 277 - 286.

[4] Whitham G B. *Linear and Nonlinear Wave*[M]. New York: John Wiley and Sons, Inc, 1974.

[5] Zhang H M. Analyses of the stability and wave properties of a new continuum traffic theory[J]. *Transp Res B*, 1999, **33**(6): 399 - 415.