Numerical Study on Traffic Flow with Single Parameter State Equation

Zuo-Jin Zhu[1]; Qing-Song Wu[2]; Rui Jiang[3]; and Tong-Qiang Wu[4]

Abstract: Traffic flow has been studied numerically by solving the kinematic wave equation with the second-order Monotone Upwind Scheme of Conservation Law (MUSCL), together with the boundary and initial conditions, which are examined by a computer based random generator derived from the Erlang process of order 250. With regard to traffic mixing, a fundamental flow-density diagram of road traffic is presented, where the ratio between the optimal and jam densities is used as a single parameter; its value is predicted by assuming that fast moving vehicles have a relatively large free speed but slow moving vehicles have a smaller free speed. Simple analysis for the state equation indicates that the parameter should be in a proper range from 0.333 to 0.618 to ensure a free speed beyond the optimal traffic speed. The effects of the single parameter on the spread of traffic shock wave have been discussed. It is found that, for congested traffic flow, in the case of a given flow density at the place of inlet and exit, the effects of the parameter on the propagation speed is apparent, while in the case of assigned flow rate on the inlet and the exit boundaries, the propagation speed is slightly dependent on the parameter. The propagation of density and flow rate fluctuation can be observed clearly from the corresponding 3D presentations.

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Introduction

It is well known that the kinematic wave model based upon the hydrodynamic analogy to traffic flow was developed by Lighthill and Whitham (1955) and Richards (1956) independently. As reported by Haight (1974), the generalizations of the Lighthill-Whitham-Richards theory (LWR) have been carried out by De (1956) and Bick and Newell (1960). Despite its history of nearly half a century, the kinematic wave model is still used extensively in current traffic research and transportation engineering. To simulate the congested traffic flow, the finite difference approximation of the kinematic wave equation was given using the Godunov scheme by Bui et al. (1992). Several years later, using the receiving and sending function, Daganzo (1995a, b) reported a similar approach of approximation that can be shown to be a member of Godunov family, but of the first order, to capture the traffic shock wave. In addition, the development of traffic flow study in the direction of the high-order model initiated by Payne (1971) and the research done by Michalopoulos et al. (1984) should be noted. However, the arguments proposed by Daganzo (1995a, b) have caused much more attention.

Particularly, the application of LWR theory requires the fundamental diagram to be expressed by a state equation. In this paper, a relation for the fundamental diagram with respect to the ratio $[b=(\rho_o/\rho_m)]$ of optimal density to that of a jam has been proposed. Based on that, the LWR equation has been solved by using the second-order Monotone Upwind Scheme of Conservation Law (MUSCL) [i.e., that of Van Leer (1979), cf. Shui (1998)] to simulate the traffic flow. The density ratio for the traffic flow is examined in the intuitive sense that a fast moving vehicle has a relatively large ratio of free speed V_f to the optimal speed V_o. It is found that the density ratio has a significant impact on the propagation of the traffic shock wave.

The mixing performance of the traffic flow is reflected by taking into account the inlet and exit flow reflecting the random properties. When fast, medium, and slow vehicles travel on the same road, the space gap of the consecutive vehicles is suggested to be consistent with an Enlang process of order 250. This is implemented by a computer-based random generator.

Single Parameter State Equation for Traffic

The fundamental diagram can be expressed in terms of a state equation. Haight (1974) delineates seven types of state equation, some of which were derived from the car following model, while others were obtained from statistical data analysis. These are listed in Table 1. However, for mixing traffic flow, none of these is directly applicable without appropriate consideration of the mixing performance. Therefore, from the theoretical point of view, it is important to propose a state equation regarding the behavior of different vehicles on the road. Taking the ratio of optimal to jam density as a parameter, it is easy for one to see the following equation with respect to both empty and jam road situations:

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Table 1. Existing Traffic Flow-Density Relationships[a]

Case	Expression	Proof	Parameter constraints
I	$q=C(1-\rho/\rho_m)$	Car following	None
II	$q=C\rho\ \ln(\rho_m/\rho)$	Car following	$C=V_f$
III	$q=V_f\rho e^{-C_\rho}$	Car following	$C=1/\rho_o$
IV	$q=[V_f\rho\ \ln(\rho_m/\rho)/C+C_1\ln(\rho_m/\rho)]$	Statistical	When $C=C_1$, C, C_1 Vanish
V	$q=[C_1\rho(\rho_m-\rho)/C_2\rho_m+C_\rho]$	Statistical	$C_1=C_2V_f$
VI	$q=\rho V_f(1-\rho/\rho_m)$	Empirical	None
VII	$q=[\rho V_f(\rho_m-\rho)^{1/2}/CV_f\rho^2+(\rho_m-\rho)^{1/2}]$	Empirical	None

[a]Abstracted from Haight (1974); some appropriate changes have been made, e.g., $m_0\to V_f$, $\rho\to q$, $\lambda\to\rho$, etc.

$$\frac{q}{q_o}=1+A(b)\left(\frac{\rho-\rho_o}{\rho_m}\right)^2+B(b)\left(\frac{\rho-\rho_o}{\rho_m}\right)^3 \quad (1)$$

where q_o and ρ_m are, respectively, the optimal flow capacity and the jam density; while $b[=(\rho_o/\rho_m)]$ is the single parameter; and $A(b)$ and $B(b)$ are functions of b. By setting $\rho=0$ and $\rho=\rho_m$, respectively, and remembering that traffic rate q should vanish at the limits of ρ, it follows that

$$A(b)b^2-B(b)b^3=-1$$
$$A(b)(1-b)^2+B(b)(1-b)^3=-1 \quad (2)$$

The solution of the foregoing equation for $A(b)$ and $B(b)$ is given by

$$A(b)=-[(1-b)^3+b^3]/[b^2(1-b)^2]$$
$$B(b)=+[(1-b)^2-b^2]/[b^2(1-b)^2] \quad (3)$$

If the traffic flow rate and the density are normalized by the optimum flow capacity and the jam density, respectively, for convenience, by use of normalization $q/q_o\to q$ and $\rho/\rho_m\to\rho$, the normalized equation of traffic state can be written as

$$q(\rho,b)=1+A(b)(\rho-b)^2+B(b)(\rho-b)^3 \quad (4)$$

The derivative with respect to ρ of this equation at $\rho=0$ is the free speed of a vehicle in the unit of q_o/ρ_m (say, α, which is the ratio between free speed and optimal speed), since

$$\alpha=\frac{V_f}{V_o}=\frac{\rho_o}{\rho_m}\lim_{\rho\to 0}\frac{q(\rho,b)}{\rho}=b\frac{d}{d\rho}q(\rho,b) \quad (5)$$

That means

$$\alpha=-2A(b)b^2+3B(b)b^3=(2-3b)/(1-b)^2 \quad (6)$$

where the expressions for $A(b)$ and $B(b)$ have been substituted. This equation is identical to

$$\alpha b^2+(3-2\alpha)b+\alpha-2=0 \quad (7)$$

which can be observed as a quadratic equation for b. Only its positive real root of this equation attains the physical meaning. Clearly, it is given by

$$b=[-(3-2\alpha)+\sqrt{9-4\alpha}]/2\alpha \quad (8)$$

where the determinant of Eq. (5) requires $\alpha\leqslant 9/4$. Thus, the corresponding value for b should be 1/3. In practical situations, the ratio α between free traffic speed and the optimal speed should be greater than unity. In view of this reality for traffic flow, at this limit, we have

$$b=(-1+\sqrt{5})/2=0.618 \quad (9)$$

It is assumed that such a value of b with respect to $\alpha=1$ is relevant to the slower moving vehicles. But the other pair of value $b=1/3$ and $\alpha=9/4$ is for the fastest vehicles. This implies a fast vehicle can run more quickly than a slow one on an empty road. Fig. 1 conveys the fundamental diagram for different density ratios.

Consider a traffic flow which consists of N types of vehicles with the ith free speed and relevant proportion to be $V_{f,i}$ and P_i, respectively. Further denote the ith ratio for free to optimal speed as α_i, which is in the range [1,9/4]. Then if the traffic is homogeneously mixed with various type vehicles, the velocity ratio for the mixed vehicles can be expressed by

$$\alpha=\sum_{i=1}^{N}\alpha_iP_i \quad (10)$$

Substituting it into Eq. (8), the single parameter for fundamental diagram can be obtained. However, this is just a compromise strategy for the determination of α, since in practice the homogeneous traffic state on empty road needs arguments.

Kinematic Wave Model

The kinematic wave model in the Lighthill-Whitham-Richards (LWR) theory gives the dimensionless equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0 \quad (11)$$

which satisfies the conservation law, where ρ and q=mean density and mean flow rate; and t and x=dimensionless time and space normalized by $1/q_o$ and $1/\rho_m$, respectively. For simplicity, it is assumed that the road segment is homogeneous, and that

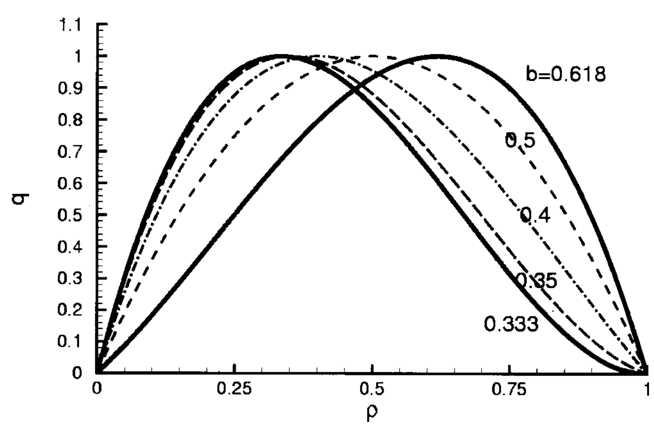

Fig. 1. Traffic flow-density diagram for several values of parameter b

vehicular interactions such as passing and overtaking have no impact on the vehicular proportions in the traffic flow.

However, the kinematic wave [Eq. (11)] must be solved numerically together with the required initial and boundary conditions, which can reflect a certain fluctuating performance of traffic flow. Such conditions are obtained by using a numerical random generator based on the Erlang process (Haight 1974) of order 250. Since the probability distribution of the Erlang process corresponds to the sum of 250 negative exponential gaps, it is not difficult to construct a computer-based random number generator. By choosing the total number of random trails to be 250, and by truncating the lower and upper locations for gaps at $\tau_l = (1/3\rho)$ and $\tau_u = (3/\rho)$, the inlet and exit densities for our numerical boundary conditions can be examined. The results are shown in Fig. 2, where the fluctuated performance of traffic density is evident.

Numerical Scheme

For convenience, the governing equation, Eq. (11), is discretized on a uniform time space grid system. Define n as the time level, with j as the space node. Further let $a(\rho) = (dq/d\rho)$ denote the propagation speed of small perturbations in traffic flow, while $\Delta\rho_{j+1/2} = \rho_{j+1} - \rho_j$ denotes the density difference between the consecutive nodes, and $\delta\rho_j$ is

$$\delta\rho_j = \begin{cases} S \min[|\Delta\rho_{j+1/2}|, |\Delta\rho_{j-1/2}|] & \text{if } S = \text{sgn}\,\Delta\rho_{j+1/2} = \text{sgn}\,\Delta\rho_{j-1/2} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where S=sign function. Then, if $\lambda = \Delta t/\Delta x$, where Δt and Δx are the time and space intervals, the MUSCL scheme proposed by Van Leer (1979) gives the density at the new time level as

$$\rho_j^{n+1} = \rho_j^n - \lambda(\hat{q}_{j+1/2} - \hat{q}_{j-1/2}) \tag{13}$$

with

$$\hat{q}_{j+1/2} = \begin{cases} \frac{1}{2} q(\rho_{L,j+1/2}) + \frac{1}{2} q\left[\rho_{L,j+1/2} - \frac{\bar{a}_{j+1/2}\lambda\delta\rho_j}{1+\lambda(\bar{a}_{j+1/2} - \bar{a}_{j-1/2})}\right] & \text{if } \bar{a}_{j+1/2} > 0 \\ \frac{1}{2} q(\rho_{R,j-1/2}) + \frac{1}{2} q\left[\rho_{R,j+1/2} - \frac{\bar{a}_{j+1/2}\lambda\delta\rho_{j+1}}{1+\lambda(\bar{a}_{j+3/2} - \bar{a}_{j+1/2})}\right] & \text{if } \bar{a}_{j+1/2} \leqslant 0 \end{cases} \tag{14}$$

where $\bar{a}_{j+1/2}$ is expressed by

$$\bar{a}_{j+1/2} = \begin{cases} \frac{q(\rho_{R,j+1/2}) - q(\rho_{L,j+1/2})}{\rho_{R,j+1/2} - \rho_{L,j+1/2}} & \text{if } \rho_{R,j+1/2} \neq \rho_{L,j+1/2} \\ a(\rho_{L,j+1/2}) & \text{otherwise} \end{cases} \tag{15}$$

with

$$\rho_{R,j+1/2} = \rho_{j+1}^n - \tfrac{1}{2}\delta\rho_{j+1} \qquad \rho_{L,j+1/2} = \rho_j^n + \tfrac{1}{2}\delta\rho_j \tag{16}$$

The finite-difference approximation to the kinematic wave equation is a second-order member of the Godunov hierarchy, from which it is easy to see that the propagation speed $a(\rho)$ for a small disturbances (see the stereo-view in Fig. 6) is closely related to the fundamental diagram. An arbitrary choice of such a diagram could not indicate the mixing performances of traffic. However, detailed consideration from the theoretical point of view as provided in the preceding section shows that the single parameter relationship (1) for the traffic state may be a reasonable alternative. In addition, what is required for the application of the numerical scheme is only the parameter λ, rather than assigning of Δt and Δx, respectively. It is important to note that the choice of λ should be less than unity to satisfy the Courant-Friedrichs-Lewy condition, e.g., $|a_{\max}\Delta t/\Delta x| \leqslant 1$.

Results and Discussions

What are the patterns of the mixing traffic flow? Such questions will be discussed by virtue of the numerical results obtained in the case of congested traffic, since it is more attractive. First, to demonstrate clearly the influence of the single parameter b on the fundamental diagram on the traffic shock wave, we consider an ideal situation, under which the random properties of arrival and departure of vehicles on the considered road segment have been neglected.

Fig. 3(a) depicts the shock wave shape and its location at the moment $t = 70\Delta t$ for several values of b assigned when the inlet density $\rho_1 = 0.2$, the exit density $\rho_2 = 0.8$, and the jump of the initial solution is at the place of $x = 30\Delta x$. An inspection of the density curves by comparing shock wave position for various values of b when $t = 70\Delta t$ with the initial one indicated that there

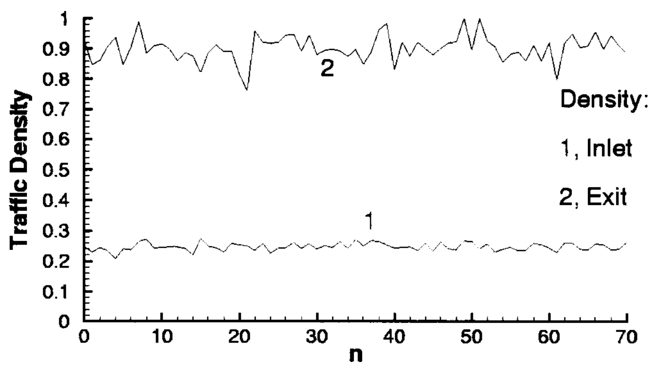

Fig. 2. Inlet and exit traffic densities predicted by computer-based random generator, which was constructed in terms of Erlang process of order 250

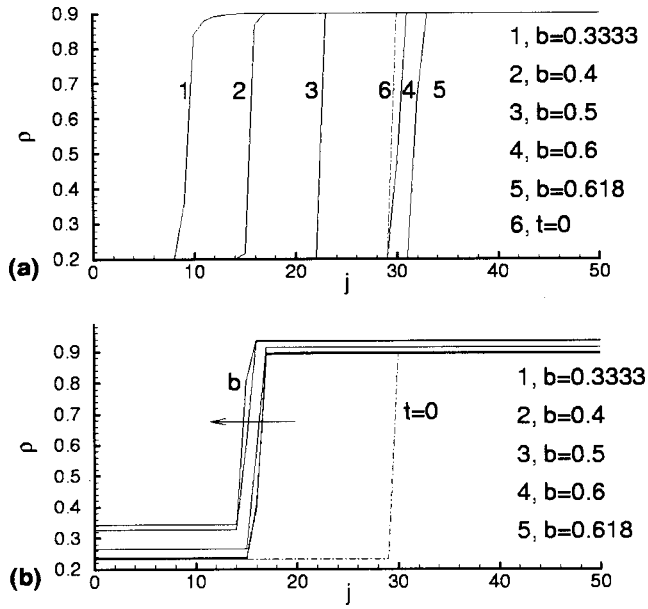

Fig. 3. Traffic shock wave shapes and locations at time $n=t/\Delta t=70$ for several values of single parameter, where: (a) for $\rho_1=0.2$, $\rho_2=0.8$; (b) for $q_1=0.8$, $q_2=0.3$; and $j=x/\Delta x$

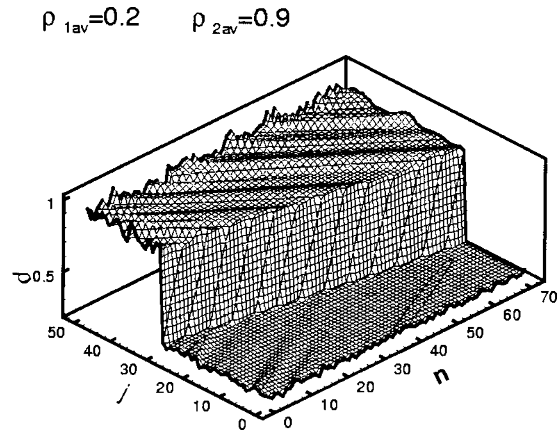

Fig. 4. 3D presentation of traffic flow density, where $j=x/\Delta x$, $n=t/\Delta t$

were two types of traffic shock waves, with one propagating backward corresponding to the relatively small value of b, and the other propagating forward for the larger value of b. This verified that the foregoing assumption (vehicles moving fast would have a relatively large ratio between the free speed and the optimal speed) does reflect a certain reality of traffic flow, since, under an actual congested flow situation, on a given road, the faster the vehicles run, the faster the shock wave backward spreads.

The backward propagation implies that a serious delay of arrival for drivers in upstream will happen. In fact, the direction of shock wave propagation is dependent on the fundamental diagram ascribed by the corresponding state equation. For, according to the (ρ,q) curve characterized by the single parameter b, the slope of the line linked from (ρ_1,q_1) to (ρ_2,q_2) can be obtained. Since the value of slope is the speed of shock wave spreading, the sign of the slope must give rise to the direction of the shock wave traveling, i.e., backward or forward.

However, in the cases of fixed inlet flow rate $q_1=0.8$ and exit flow rate $q_2=0.3$ [Fig. 3(b)], the effects of parameter b on the traffic shock is not as significant as in the cases for fixed densities at the same places. In fact, Fig. 3(b) indicates that the shocks at the moment $t=70\Delta t$ for $b=0.333$ and $b=0.4$ seem to be overlapped. All shocks in this cases traveled backward, with those for the large value of b spreading slightly faster. It is clear that different shock wave shapes can be obtained from different choices of b.

Subsequently, after discussing the effects of parameter b, attention is concentrated on the patterns of traffic flow in mixing. In practice, the traffic flow is closed related to the microbehaviors of drivers, the condition of the road, the climate conditions, and so on. The treatment of vehicular mixing, as mentioned in the preceding sections, is carried out with the following procedure:

1. Consider the flow to be constituted with several groups of vehicles distinguished by free traffic speed $V_{f,i}$, $i=1,2,\ldots,N$; the sequence takes the order decreased in i;
2. Assign a set of b_i corresponding to $V_{f,i}$, and let b_i take the inverse order of $V_{f,i}$ with respect to the postulation that fast moving implies a small value of the single parameter b;
3. Calculate α_i with respect to b_i from Eq. (6), where b and α should be replaced by b_i and α_i; and
4. Calculate the speed ratio between the free speed and the optimal speed α from Eq. (10). By using Eq. (8), the value of b can be obtained.

The calculated α_i and the value of b have been summarized in Table 2. From Table 2, when the total group number $N=6$, and the binomial parameter $p=0.4$, it follows that α and b are 2.126 and 0.460, respectively. The results illustrated in Figs. 4 and 5 are obtained with these values.

The 3D presentation of the traffic density of flow in the (x,t) plane is shown in Fig. 4. In simulation, the ratio $\Delta t/\Delta x=0.25$ means the Courant-Friedrichs-Levy condition for the choice of time interval was satisfied; since the stereo-view in Fig. 6 indicates that the absolute value of maximum wave speed is less than 4.0, the corresponding Courant number is near 0.9. It is seen that, in this case, when 70 time intervals have elapsed, the shock has propagated backward for about 17 meshes. This implies that the wave speed of propagation is about -0.243, i.e. the actual wave speed is about $-(17q_o/70\rho_m)$. If the jam density for the traffic flow is $\rho_m=86$ vh/km (here, vh denotes vehicle, while km denotes

Table 2. Parameter b for Mixing Traffic with b_i Assigned

Speed ratio α_i corresponding to b_i assigned for vehicles with different $V_{f,i}$							Value of b with respect to α_i, $V_{f,i}$, and P_i given by $P_i=\{(N-1)!/[(i-1)!(N-i)!]\}p^{i-1}(1-p)^{N-i}$, $i=1,2,...,N$			
$V_{f,i}$	70 km/h	60 km/h	50 km/h	40 km/h	35 km/h	30 km/h	N	p	α	b
b_i	0.333	0.4	0.45	0.5	0.55	0.618	6	0.2	2.210	0.411
α_i	2.250	2.222	2.149	2.000	1.728	1.000	6	0.3	2.176	0.436
							6	0.4	2.126	0.460

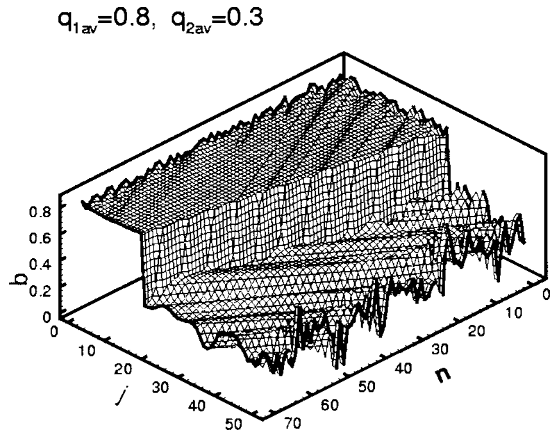

Fig. 5. 3D presentation of traffic flow rate, where $j=x/\Delta x$, $n=t/\Delta t$

kilometer) with a road optimal capacity $q_o=1{,}440$ vh/hr, the wave speed is about 4.066 km/hr (hr denotes hour). One advantage of the 3D presentation is that it can clearly demonstrate the behavior of wave spread. Fig. 5 shows the traffic flow rate distribution on the urban road segment. The fluctuating flow rate implies that an observer standing beside the road at a fixed position will have insight of a view of maneuver patterns of vehicles; they are traveling quickly at one time, but slowly at another.

Conclusions

In this work, the traffic flow on a road segment with shock wave has been studied by virtue of a single parameter state equation for the relation between the flow rate and density. The parameter, which is the ratio between optimal density and jam density, is closely related to the ratio between the free speed and the optimal one. Simple analysis indicated that the parameter should be in the range between 0.333 and 0.618. For traffic flow with mixing, it is dependent upon the free speed, the vehicular proportions, and the velocity ratio of different vehicles. The impacts of such a parameter on the spread of traffic wave has been discussed merely for an ideal situation, under which the mixing effects are completely neglected.

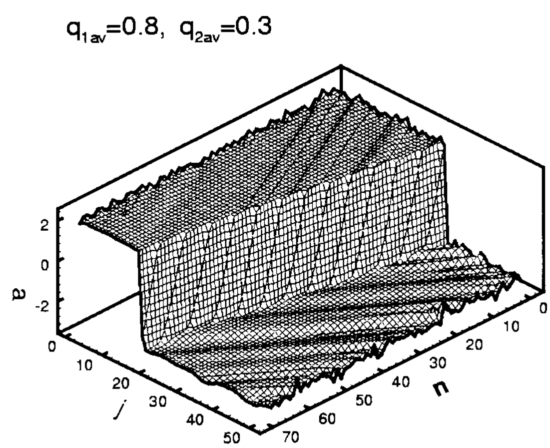

Fig. 6. 3D presentation of wave propagation speed, where $j=x/\Delta x$, $n=t/\Delta t$

The LWR model was used in the study, with the initial and boundary conditions examined by using a computer-based random generator. This random generator was constructed with respect to the Erlang process of order 250. It was found that the fluctuation in such a condition could clearly illustrate the behavior of shock propagation in the considered traffic system.

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Notation

The following symbols are used in this paper:

$A(b)$ = function of b in state equation;
a = wave propagation speed;
$\bar{a}$ = wave propagation speed;
$B(b)$ = function of b in state equation;
b = single parameter;
b_i = parameter for ith vehicular group;
N = total number of vehicular group;
P_i = proportion for ith vehicular group in mixing traffic flow;
p = parameter used to define P_i in binomial distribution;
q = normalized traffic flow rate;
S = sign;
t = time;
V_f = free vehicular speed;
$V_{f,i}$ = free vehicular for ith group;
x = space;
α = ratio between optimal free speed and optimal one;
α_i = ratio between optimal free speed and optimal one for ith vehicular group;
Δt = time interval;
Δx = space interval;
λ = ratio between space and time interval;
ρ = normalized traffic density;
ρ_m = jam density of traffic flow;
ρ_o = optimal density of traffic flow;
τ_l = $1/(3\rho)$, lower truncation limit; and
τ_u = $3/\rho$, upper truncation limit.

Subscripts

f = free;
i = vehicular group number;
j = space grid number;
L = left;
l = lower;
m = jam;
o = optimal; and
u = upper.

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