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Numerical study of mixed freeway traffic flows

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SUMMARY

This study presents some features in the numerical modelling of multi-class mixed freeway traffic described by extended LWR model. The evaluation of the eigenvalues and the right characteristic matrix of the mixed traffic system was reported in detail. To some extent, this is the key step in the application of the second-order symmetrical total variation diminishing (TVD) numerical method to traffic modelling. Examples for four-class mixed traffic flows were given to illustrate the solution feasibility. It was found that, when the global density wave keeps the same form, the variation of vehicular composition in the initial density wave has no effect on the mean speed of propagation of traffic wave on freeways, but can affect the decay rate of the density wave in different way. Traffic mixing and the impedance of the slow moving vehicles are reasons of the small amplitude density oscillations downstream the main global density wave. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: mixed traffic flows; characteristic matrix; TVD method; local heterogeneity

1. INTRODUCTION

The operation of transportation system depends on traffic flow, for which, Lighthill and Whitham [1] and Richards [2] have developed a model based on vehicular conservation. This model is usually called LWR model that can predict the travelling of traffic shock waves on freeways. It has been extended to describe multi-class traffic by Wong *et al.* [3]. The extended LWR model was found to be able to show the pattern of traffic hysteresis. LWR type models require a traffic state equation that relates the flow to density, and should be evaluated from careful traffic observation.

Using a single parameter state equation of mixed traffic flow, Zhu *et al.* [4] have recently found that mixed traffic flow is dependent on the free flow speed, the vehicular proportions and

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the velocity ratio of different vehicles. For freeway traffic flow, Zhu and Wu [5] have reported a two-fluids model, and found that the mixing of traffic flow provides a mechanism of traffic density wave production. Zhu, Chang and Wu [6] have further studied the mixing induced traffic oscillation of freeway traffic using a three-class model. Lebacque [7] has developed a two-phase bounded-acceleration traffic model. One phase is traffic equilibrium: flow and speed are functions of density, and traffic acceleration is low. The second phase is characterized by constant acceleration. This two-phase model also extends the LWR model and recaptures the fact that acceleration of traffic is bounded. It admits analytical solutions for most situations in which the LWR model exhibits them.

Zhang *et al.* [8] have used the WENO method [9] to study mixed traffic flow dynamics. However, some features in the numerical solution process such as how to evaluate the characteristic values were omitted.

The objective of this paper is to detail some features in the numerical modelling of mixed freeway traffic flow, highlight the evaluation of the eigenvalues and the right characteristic matrix. To some extent, this is the key step in the application of the second-order total-variation-diminishing (TVD) method [10] to the modelling of mixed freeway traffic flow. Particular four-class mixed traffic flows were studied numerically as examples of the method application.

2. THE GOVERNING EQUATIONS

Similar to the reported work [6], the mixed traffic flow containing m classes of drivers satisfies the following assumptions: (1) The ramp flow effect on the main traffic stream on the freeway can be totally excluded; (2) The vehicular responses to the traffic condition can be governed by the global density, the free-flow speed, the jam density, and most importantly the unique speed–density relation.

Let the mixed traffic flow scale with the road capacity q_0, and the relevant density scale with the jam density k_{jam}. If the length of the road segment L_0 is chosen as the scale of length, the time and speed should be scaled by $t_0 = L_0 k_{\text{jam}}/q_0$, and q_0/k_{jam}, respectively. Hence, the normalized governing equations for mixed freeway traffic flows can be written as

$$\frac{\partial k_i}{\partial t} + \frac{\partial (u_i k_i)}{\partial x} = 0, \quad i = 1, \ldots, m \tag{1}$$

where t and x have been normalized by t_0 and L_0. Under the assumption of multi-class mixed freeway traffic flows, the non-dimensional governing equation (1) is supplemented by

$$u_i = v_{f,i}(1 - k^{n_i}), \quad i = 1, \ldots, m \tag{2}$$

where $k = \sum_{i=1}^{m} k_i$, $v_{f,i}$ and n_i are the free-flow speed and the index of the speed–density relation for the ith class of vehicles. When there is a deterministic road capacity, the index value for ith class of vehicles should be a function of the optimal density, at which, the traffic flow is just equal to the road capacity. The following relation between the index and the optimal density (b_i) has been suggested in Reference [6]:

$$b_i^{n_i} = 1/(n_i + 1), \quad i = 1, \ldots, m \tag{3}$$

This relation (3) was derived by seeking the maxima of the partial flow $q_i(=k_iu_i)$ of the ith class of vehicles, taking the relevant free speed $v_{f,i}$ as constant. With respect to the single state equation given in Reference [4], an analysis has indicated that it is reasonable for the normalized optimal density b_i should be ranged from $\frac{1}{3}$ to 0.618.

Equation (2) to some extent reflects the behaviour coupling between different class of vehicles. It was postulated with respect to the one parameter supplemental relationship for single class traffic flows as mentioned in Reference [11]. It is noted that the partial speed u_i is an explicit function of the global density k and the partial free speed $v_{f,i}$. In particular, the speed u_i linearly depends on the global density when the index n_i is identical to unity. For the mixed traffic flows governed by Equation (1), using Equation (2), numerical study is possible to reveal some important performance in mixed freeway traffic.

To seek the numerical solution of the governing equations, the first and second homogeneous boundary conditions were used. The TVD method having second-order accuracy [10] was used. In this application, the key problem is to evaluate the eigenvalues and the right characteristic matrix which will be detailed in the next section.

3. SOME DETAILS IN SOLUTION PROCESS

The vector form of the governing equations can be written as follows:

$$\frac{\partial \mathbf{K}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{K})}{\partial x} = 0 \tag{4}$$

where $\mathbf{K} = (k_1, k_2, \ldots, k_m)^{\mathrm{T}}$ is the density vector, and $\mathbf{F}(\mathbf{K}) = (F_1, F_2, \ldots, F_m)^{\mathrm{T}}$ is the flow flux vector, with the superscript T denoting the matrix transposition. The Jacobian matrix for Equation (4) is given by

$$\mathbf{A} = (a_{ij}), \quad i, j = 1, \ldots, m \tag{5}$$

with element $a_{ij} = \partial F_i / \partial k_j$. As reported by Wang *et al.* [12], using the Faddeev–Leverrier method [13], the characteristic equation can be written as

$$\lambda^m - \sum_{i=1}^{m} p_i \lambda^{m-i} = 0 \tag{6}$$

with

$$\begin{aligned}
& p_1 = \operatorname{tr} \mathbf{A}, \quad \mathbf{A}_1 = \mathbf{A}(\mathbf{A} - p_1 \mathbf{I}) \\
& p_2 = \tfrac{1}{2} \operatorname{tr} \mathbf{A}_1, \quad \mathbf{A}_2 = \mathbf{A}(\mathbf{A}_1 - p_2 \mathbf{I}) \\
& \ldots, \ldots \\
& p_{m-1} = \frac{1}{m-1} \operatorname{tr} \mathbf{A}_{m-2}, \quad \mathbf{A}_{m-1} = \mathbf{A}(\mathbf{A}_{m-2} - p_{m-1} \mathbf{I}) \\
& p_m = \frac{1}{m} \operatorname{tr} \mathbf{A}_{m-1}
\end{aligned} \tag{7}$$

where $\mathbf{I}$ is the unit matrix; while tr means the trace of a matrix. For example, the trace of the characteristic matrix $\mathbf{A}$ is the summation of the diagonal elements given by $\sum_{i=1}^{m} a_{ii}$. It can be proved that the inversion of the matrix $\mathbf{A}$ can be rewritten as $\mathbf{A}^{-1} = (\mathbf{A}_{m-2} - p_{m-1}\mathbf{I})/p_m$.

The eigenvalues of the governing equation (4) are the roots of the characteristic equation (6). When the speed–density relation (2) is used, the eigenvalues are all real. Labelling with an increasing order, one obtains

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m \tag{8}$$

For larger m, the eigenvalues can be evaluated by using Newton's iteration method [14]. However, for the situation of $m = 4$, according to Reference [15], the eigenvalues can be evaluated with the following procedure. The roots of Equation (6) are identical to the roots of the pair of quadratic equations

$$\lambda^2 + \tfrac{1}{2}(-p_1 \pm [8y + p_1^2 + 4p_2]^{1/2})\lambda + (y \pm (-p_1 y + p_3)[8y + p_1^2 + 4p_2]^{-1/2}) = 0 \tag{9}$$

where y are an arbitrary real root of the cubic equation

$$y^3 + \tfrac{1}{2}p_2 y^2 + \tfrac{1}{4}(p_1 p_3 + 4p_4)y + \tfrac{1}{8}[p_4(4p_2 + p_1^2) - p_3^2] = 0 \tag{10}$$

The solutions of Equation (10) can be sought with the method used in the reported work [6].

For the eigenvalues of mixed freeway traffic flow, there exists a right characteristic matrix with the form

$$\mathbf{R} = [\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}] = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{pmatrix} \tag{11}$$

The matrix elements can be obtained in terms of Gauss elimination approach [14], with the left characteristic matrix $\mathbf{L}(= \mathbf{R}^{-1})$ obtained using the coefficients of matrix $\mathbf{R}$ given by the Faddeev–Leverrier method [13].

After evaluating matrices $\mathbf{R}$ and $\mathbf{L}$, the TVD method [10] with second-order accuracy can be applied conveniently to carry out the numerical modelling of mixed traffic flow on freeways.

It is expected these features in the solution process are invaluable in the numerical study of other physical and engineering problems, such as in gas-dynamics and multiphase fluid dynamics, where the numerical method TVD also has its market of application.

4. RESULTS AND DISCUSSION

This section presents examples in the numerical modelling of the particular four-class mixed freeway traffic flows described by the extended LWR model. Initial parameters for performing the numerical analyses are summarized below:

- The free-flow speed of ith class is given by the expression

$$v_{f,i} = 1/[b_i(1 - b_i^{n_i})], \quad i = 1, \ldots, 4$$

- The optimal density values are assigned as $b_i = 0.4, 0.41, 0.42,$ and 0.43, for which, the parameters of speed–density relation (n_i, $i = 1, 2, 3, 4$) for these four driving classes can be evaluated using Equation (3);

- The total length of the freeway segment was assumed as 20 km, and used as the length scale in the present simulation. The road capacity q_0 and the jam density k_{jam} were assumed as 1800 vh/h and 120 vh/km, respectively.

Initial conditions given in Figures 1(a) and (b) were used in the modelling process. Figure 1(a) shows two density square waves arisen from the local heterogeneity of the 1st or 4th class of vehicles. The initial density distributions were represented, respectively, by the dashed and solid lines. It was assumed that the wave has a magnitude 0.01, and occurs in the range of $x \in (0.2, 0.3)$. The densities of the 2nd and the 3rd class of vehicles are 0.09 and 0.06, respectively, and have uniform initial distributions. Hence, outside the wave region, the global density is about 0.3. While Figure 1(b) includes two density square waves with the same magnitude, but occurs in the range $x \in (0.7, 0.8)$. For this case, the values of the 2nd and 3rd classes of vehicles are, respectively, 0.15 and 0.1, and also distribute uniformly. Thus, outside the wave region, the global density is 0.5.

Since the optimal densities of the mixed traffic for the four classes are, respectively, 0.4, 0.41, 0.42, and 0.43, the initial wave shown in Figure 1(a) should propagate forward, but the wave shown in Figure 1(b) should travel backward.

Figure 2(a) shows that the density wave front initially positioned at $x = 0.3$ has arrived at about $x = 0.7$ at the instant of $t = 0.306$. The mean speed of the density wave propagation during the initial period is approximately identical to 13.07 km/h ($= (0.7 - 3)/0.306 \times q_0/k_{\text{jam}}$). Such a value is in the speed range of traffic waves on freeways: from 10 to 20 km/h, as noted by Helbing and Treiber [16] from the empirical findings of Kerner and Rehborn [17, 18]. It should be noted that this particular consistency does not imply that the traffic wave propagation speed less than 10 km/h is impractical, since from theoretical point of view, the propagation speed is dependent on the supplemented traffic state equations, or more explicitly the values of the optimal density selected. Practical traffic wave propagation speed may also closely relate to the generating sources of the traffic waves.

As indicated by the solid curves in Figure 2(a), more evident in the near region of point $x = 0.7$, the small amplitude density oscillations downstream the main global density wave are evidence of traffic mixing and the impedance of the 4th class of vehicles, but not the

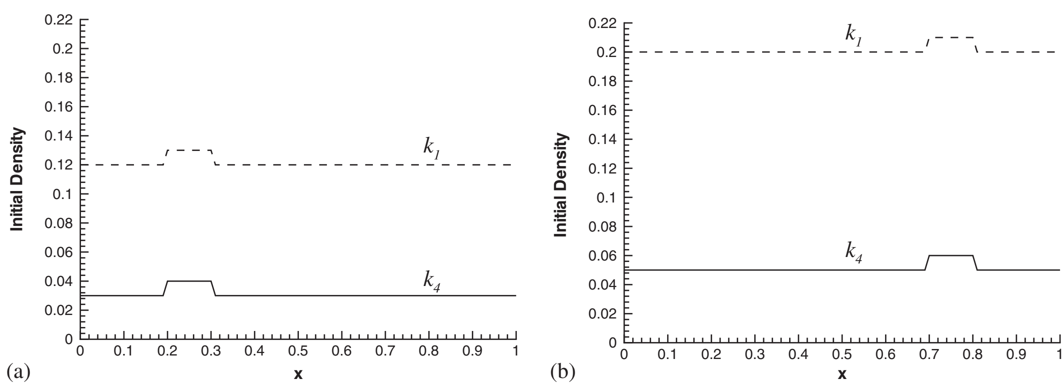

Figure 1. Density disturbance in square wave form on the initial traffic road for: (a) stable; and (b) unstable highway traffic flows. The magnitude of the square wave is 0.01, it is represented by dash or solid line as it occurs in the 1st or 4th class of vehicles.

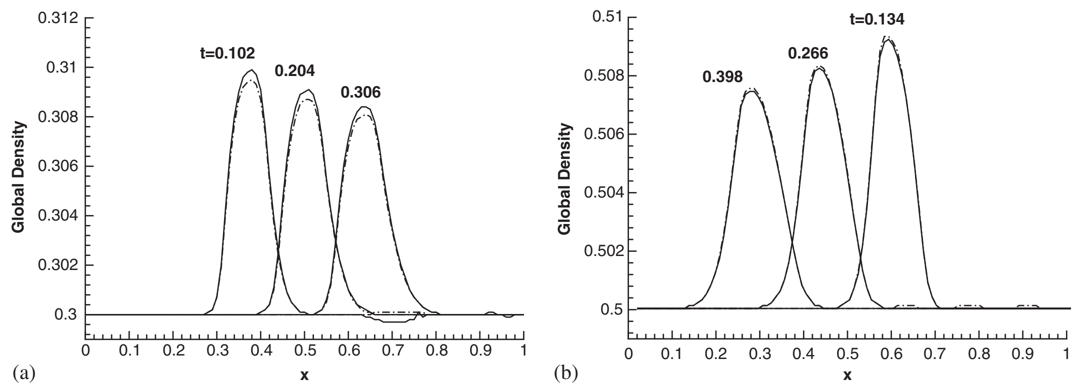

Figure 2. Density distribution for the three instants for the cases of: (a) forward; and (b) backward propagation of traffic wave. Note that the dashed curve denotes the case of the 1st class disturbance in the initial field, with the solid curve representing the case of the 4th class disturbance in the initial field.

implication of numerical instability. These small density oscillations should be pronounced when the initial density is less the optimal density of the freeway traffic [6].

Generally, the wave evolution occurs a decaying trend. By comparison with the dashed and solid curves given in Figure 2(a), it was seen that, during the initial period, the mean travelling speed of the density wave is rigid to the variation of vehicular composition in the initial wave, or, in other words, without regard to the initial wave is arisen from the local heterogeneity of the 1st or the 4th class of vehicles. When the initial wave is arisen from the 4th class local heterogeneity, as shown by the solid curve in Figure 2(a), at the same instant, the wave magnitude is higher, implying the presence of a lower decaying rate. Furthermore, as indicated by the solid curve in the range of $x > 0.6$, there is also a comparatively significant influence on the downstream density uniformity.

In contrast to what was seen in Figure 2(a), Figure 2(b) shows that the traffic wave decays a bit more rapidly when the initial density wave is arisen from the 4th class local heterogeneity. Similarly, the wave front given by the solid curve has been overlapped by the dashed curves, indicating that again the mean speed of the backward traffic wave during the initial period is independent of the variation of vehicular composition in the initial wave. For this case, the absolute value of the mean speed of wave propagation is about 18.09 km/h ($= |0.7 - 0.22|/0.398 \times 15$ km/h), which are also coincident with the empirical findings of Kerner and Rehborn [17, 18].

5. CONCLUSION

This paper has detailed some features in the numerical modelling of multi-class mixed traffic flow model described by extended LWR model. The evaluation of the eigenvalues and the right characteristic matrix has been presented in detail. Numerical solutions based on TVD method indicate that, for the cases of the initial global density close to 0.3 and 0.5, when the traffic is composed of four-class of vehicles, and the initial density square wave is arisen from the local heterogeneity of the slowest vehicular class in mixed freeway traffic, comparatively significant effect on the downstream uniformity occurs and the decay rate of the density wave were lower when the wave propagates forward, but a higher decay rate appears when the wave travels backward. Reverse wave decaying trend exists when the initial density square wave is arisen from the fastest class heterogeneity. The mean wave propagation speed is almost rigid to the variation of vehicular composition in the same initial wave. Traffic mixing and the impedance of the slow moving vehicles may cause small amplitude density oscillations downstream the main global density wave.

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