Proof of Inequality $Ee^{sZ} \le e^{s^2(b-a)^2/8}$

Theorem 1. *if Z is a random variable with $E[Z] = 0$ and $a \le Z \le b$, then for any real number $s > 0$,*

$$Ee^{sZ} \le e^{s^2(b-a)^2}/8$$

Proof. By the convexity of the exponential function, (i.e., the definition of convexity function: $f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \quad 0 \le t \le 1$)

$$e^{sz} \le \frac{z-a}{b-a}e^{sb} + \frac{b-z}{b-a}e^{sa}, \quad a \le z \le b,$$

thus,

$$\begin{aligned} Ee^{sZ} &\le E\frac{z-a}{b-a}e^{sb} + E\frac{b-z}{b-a}e^{sa} \\ &= \frac{-a}{b-a}e^{sb} + \frac{b}{b-a}e^{sa} \\ &= (1-\theta+\theta e^{s(b-a)})e^{-\theta s(b-a)}, \end{aligned}$$

where $\theta = -\frac{a}{b-a}$.

Now let $u = s(b-a)$ and define $\phi(u) = -\theta u + log(1-\theta+\theta e^u)$, then we have

$$Ee^{sZ} \le e^{\phi(u)}.$$

By Taylor expansion

$$\phi(u) = \phi(0) + \phi'(0)u + \frac{1}{2}\phi''(v)u^2, \quad \text{for some} \quad 0 \le v \le u$$

and

$$\begin{aligned} \phi'(u) &= -\theta + \frac{\theta e^u}{1-\theta+\theta e^u}, \Longrightarrow \phi'(0) = 0 \\ \phi''(u) &= \frac{\theta e^u}{1-\theta+\theta e^u} - \left[\frac{\theta e^u}{1-\theta+\theta e^u}\right]^2 \\ &= \frac{\theta e^u}{1-\theta+\theta e^u}(1 - \frac{\theta e^u}{1-\theta+\theta e^u}) \\ &= \rho(1-\rho) \le \frac{1}{4}, \end{aligned}$$

where $\rho = \frac{\theta e^u}{1-\theta+\theta e^u}$.

Hence,

$$Ee^{sZ} \le e^{u^2/8} = e^{s^2(b-a)^2/8}.$$

□