►Properties of Estimators
►Statistical Fallacies: Misconceptions, and Myths

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Asymptotic Relative Efficiency in Testing

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Asymptotic Relative Efficiency of Two Tests

Making a substantiated choice of the most efficient statistical test of several ones being at the disposal of the statistician is regarded as one of the basic problems of Statistics. This problem became especially important in the middle of XX century when appeared computationally simple but "inefficient" rank tests.

Asymptotic relative efficiency (ARE) is a notion which enables to implement in large samples the quantitative comparison of two different tests used for testing of the same statistical hypothesis. The notion of the asymptotic efficiency of *tests* is more complicated than that of asymptotic efficiency of *estimates.* Various approaches to this notion were identified only in late forties and early fifties, hence, 20–25 years later than in the estimation theory. We proceed now to their description.

Let $\{T_n\}$ and $\{V_n\}$ be two sequences of statistics based on n observations and assigned for testing the null-hypothesis H against the alternative A. We assume that the alternative is characterized by real parameter θ and for $\theta = \theta_0$ turns into H. Denote by $N_T(\alpha,\beta,\theta)$ the sample size necessary for the sequence $\{T_n\}$ in order to attain the power β under the level α and the alternative value of parameter θ. The number $N_V(\alpha,\beta,\theta)$ is defined in the same way.

It is natural to prefer the sequence with smaller N. Therefore the relative efficiency of the sequence $\{T_n\}$ with respect to the sequence $\{V_n\}$ is specified as the quantity

$$e_{T,V}(\alpha,\beta,\theta) = N_V(\alpha,\beta,\theta)\big/N_T(\alpha,\beta,\theta)\,,$$

so that it is the reciprocal ratio of sample sizes N_T and N_V.

The merits of the relative efficiency as means for comparing the tests are universally acknowledged. Unfortunately it is extremely difficult to explicitly compute $N_T(\alpha,\beta,\theta)$ even for the simplest sequences of statistics $\{T_n\}$. At present it is recognized that there is a possibility to avoid this difficulty by calculating the limiting values $e_{T,V}(\alpha,\beta,\theta)$ as $\theta \to \theta_0$, as $\alpha \to 0$ and as $\beta \to 1$ keeping two other parameters fixed. These limiting values $e^P_{T,V}$, $e^B_{T,V}$ and $e^{HL}_{T,V}$ are called respectively the Pitman, Bahadur and Hodges–Lehmann asymptotic relative efficiency (ARE), they were proposed correspondingly in Pitman (1949), Bahadur (1960) and Hodges and Lehmann (1956).

Only close alternatives, high powers and small levels are of the most interest from the practical point of view. It keeps one assured that the knowledge of these ARE types will facilitate comparing concurrent tests, thus producing well-founded application recommendations.

The calculation of the mentioned three basic types of efficiency is not easy, see the description of theory and many examples in Serfling (1980), Nikitin (1995) and Van der Vaart (1998). We only mention here, that Pitman efficiency is based on the central limit theorem (see ►Central Limit Theorems) for test statistics. On the contrary, Bahadur efficiency requires the large deviation asymptotics of test statistics under the null-hypothesis, while Hodges–Lehmann efficiency is connected with large deviation asymptotics under the alternative. Each type of efficiency has its own merits and drawbacks.

Pitman Efficiency

Pitman efficiency is the classical notion used most often for the asymptotic comparison of various tests. Under some regularity conditions assuming ►asymptotic normality of test statistics under H and A, it is a number which has been gradually calculated for numerous pairs of tests.

We quote now as an example one of the first Pitman's results that stimulated the development of nonparametric statistics. Consider the two-sample problem when under the null-hypothesis both samples have the same continuous distribution and under the alternative differ only in location. Let $e^P_{W,t}$ be the Pitman ARE of the two-sample Wilcoxon rank sum test (see ►Wilcoxon–Mann–Whitney Test) with respect to the corresponding Student test (see ►Student's t-Tests). Pitman proved that for Gaussian samples $e^P_{W,t} = 3/\pi \approx 0.955$, and it shows that the ARE of the Wilcoxon test in the comparison with the Student test (being optimal in this problem) is unexpectedly high. Later Hodges and Lehmann (1956) proved that

$$0.864 \leq e^P_{W,t} \leq +\infty,$$

if one rejects the assumption of normality and, moreover, the lower bound is attained at the density

$$f(x) = \begin{cases} 3\,(5 - x^2)/\left(20\sqrt{5}\right) & \text{if } |x| \leq \sqrt{5}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the Wilcoxon rank test can be infinitely better than the parametric test of Student but their ARE never falls below 0.864. See analogous results in Serfling (2010) where the calculation of ARE of related estimators is discussed.

Another example is the comparison of independence tests based on Spearman and Pearson correlation coefficients in bivariate normal samples. Then the value of Pitman efficiency is $9/\pi^2 \approx 0.912$.

In numerical comparisons, the Pitman efficiency appear to be more relevant for moderate sample sizes than other efficiencies Groeneboom and Oosterhoff (1981). On the other hand, Pitman ARE can be insufficient for the comparison of tests. Suppose, for instance, that we have a normally distributed sample with the mean θ and variance 1 and we are testing $H : \theta = 0$ against $A : \theta > 0$. Let compare two significance tests based on the sample mean $\bar{X}$ and the Student ratio t. As the t-test does not use the information on the known variance, it should be inferior to the optimal test using the sample mean. However, from the point of view of Pitman efficiency, these two tests are equivalent. On the contrary, Bahadur efficiency $e^B_{t,\bar{X}}(\theta)$ is strictly less than 1 for any $\theta > 0$.

If the condition of asymptotic normality fails, considerable difficulties arise when calculating the Pitman ARE as the latter may not at all exist or may depend on α and β. Usually one considers limiting Pitman ARE as $\alpha \to 0$. Wieand (1976) has established the correspondence between this kind of ARE and the limiting approximate Bahadur efficiency which is easy to calculate.

Bahadur Efficiency

The Bahadur approach proposed in Bahadur (1960; 1967) to measuring the ARE prescribes one to fix the power of tests and to compare the exponential rate of decrease of their sizes for the increasing number of observations and fixed alternative. This exponential rate for a sequence of statistics $\{T_n\}$ is usually proportional to some non-random function $c_T(\theta)$ depending on the alternative parameter θ which is called the *exact slope* of the sequence $\{T_n\}$. The Bahadur ARE $e^B_{V,T}(\theta)$ of two sequences of statistics $\{V_n\}$ and $\{T_n\}$ is defined by means of the formula

$$e^B_{V,T}(\theta) = c_V(\theta) \big/ c_T(\theta).$$

It is known that for the calculation of exact slopes it is necessary to determine the large deviation asymptotics of a sequence $\{T_n\}$ under the null-hypothesis. This problem is always nontrivial, and the calculation of Bahadur efficiency heavily depends on advancements in large deviation theory, see Dembo and Zeitouni (1998) and Deuschel and Stroock (1989).

It is important to note that there exists an upper bound for exact slopes

$$c_T(\theta) \leq 2K(\theta)$$

in terms of Kullback–Leibler information number $K(\theta)$ which measures the "statistical distance" between the alternative and the null-hypothesis. It is sometimes compared in the literature with the ►Cramér–Rao inequality in the estimation theory. Therefore the absolute (nonrelative) Bahadur efficiency of the sequence $\{T_n\}$ can be defined as $e^B_T(\theta) = c_T(\theta)/2K(\theta)$.

It is proved that under some regularity conditions the likelihood ratio statistic is asymptotically optimal in Bahadur sense (Bahadur 1967; Van der Vaart 1998, Sect. 16.6; Arcones 2005).

Often the exact Bahadur ARE is uncomputable for any alternative θ but it is possible to calculate the limit of Bahadur ARE as θ approaches the null-hypothesis. Then one speaks about the *local* Bahadur efficiency.

The indisputable merit of Bahadur efficiency consists in that it can be calculated for statistics with non-normal asymptotic distribution such as Kolmogorov-Smirnov, omega-square, Watson and many other statistics.

Asymptotic Relative Efficiency in Testing. Table 1 Some local Bahadur efficiencies

Statistic	Distribution					
	Gauss	Logistic	Laplace	Hyperbolic cosine	Cauchy	Gumbel
D_n	0.637	0.750	1	0.811	0.811	0.541
ω_n^2	0.907	0.987	0.822	1	0.750	0.731

Consider, for instance, the sample with the distribution function (df) F and suppose we are testing the goodness-of-fit hypothesis $H_0 : F = F_0$ for some known continuous df F_0 against the alternative of location. Well-known distribution-free statistics for this hypothesis are the Kolmogorov statistic D_n and omega-square statistic ω_n^2. The following table presents their local absolute efficiency in case of six standard underlying distributions:

We see from Table 1 that the integral statistic ω_n^2 is in most cases preferable with respect to the supremum-type statistic D_n. However, in the case of Laplace distribution the Kolmogorov statistic is locally optimal, the same happens for the Cramér-von Mises statistic in the case of hyperbolic cosine distribution. This observation can be explained in the framework of Bahadur local optimality, see Nikitin (1995 Chap. 6).

See also Nikitin (1995) for the calculation of local Bahadur efficiencies in case of many other statistics.

Hodges–Lehmann efficiency

This type of the ARE proposed in Hodges and Lehmann (1956) is in the conformity with the classical Neyman-Pearson approach. In contrast with Bahadur efficiency, let us fix the level of tests and let compare the exponential rate of decrease of their second-kind errors for the increasing number of observations and fixed alternative. This exponential rate for a sequence of statistics $\{T_n\}$ is measured by some non-random function $d_T(\theta)$ which is called the Hodges–Lehmann index of the sequence $\{T_n\}$. For two such sequences the Hodges–Lehmann ARE is equal to the ratio of corresponding indices.

The computation of Hodges–Lehmann indices is difficult as requires large deviation asymptotics of test statistics under the alternative.

There exists an upper bound for the Hodges–Lehmann indices analogous to the upper bound for Bahadur exact slopes. As in the Bahadur theory the sequence of statistics $\{T_n\}$ is said to be *asymptotically optimal in the Hodges–Lehmann sense* if this upper bound is attained.

The drawback of Hodges–Lehmann efficiency is that most *two-sided* tests like Kolmogorov and Cramér-von Mises tests are asymptotically optimal, and hence this kind of efficiency cannot discriminate between them. On the other hand, under some regularity conditions the one-sided tests like linear rank tests can be compared on the basis of their indices, and their Hodges–Lehmann efficiency coincides locally with Bahadur efficiency, see details in Nikitin (1995).

Coupled with three "basic" approaches to the ARE calculation described above, intermediate approaches are also possible if the transition to the limit occurs simultaneously for two parameters at a controlled way. Thus emerged the Chernoff ARE introduced by Chernoff (1952), see also Kallenberg (1982); the intermediate, or the Kallenberg ARE introduced by Kallenberg (1983), and the Borovkov–Mogulskii ARE, proposed in Borovkov and Mogulskii (1993).

Large deviation approach to asymptotic efficiency of tests was applied in recent years to more general problems. For instance, the change-point, "signal plus white noise" and regression problems were treated in Puhalskii and Spokoiny (1998), the tests for spectral density of a stationary process were discussed in Kakizawa (2005), while Taniguchi (2001) deals with the time series problems, and Otsu (2010) studies the empirical likelihood for testing moment condition models.

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Professor Nikitin is Chair of Probability and Statistics of St. Petersburg University. He is an Associate editor of *Statistics and Probability Letters*, and member of the editorial Board, *Mathematical Methods of Statistics* and *Metron*. He is a Fellow of the Institute of Mathematical Statistics. Professor Nikitin is the author of the text *Asymptotic efficiency of nonparametric tests*, Cambridge University Press, NY, 1995, and has authored more than 100 papers, in many international journals, in the field of Asymptotic efficiency of statistical tests, large deviations of test statistics and nonparametric Statistics.

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►Asymptotic Relative Efficiency in Estimation
►Chernoff-Savage Theorem
►Nonparametric Statistical Inference
►Robust Inference

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Asymptotic, Higher Order

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Higher order asymptotic deals with two sorts of closely related things. First, there are questions of approximation. One is concerned with expansions or inequalities for a distribution function. Second, there are inferential issues. These involve, among other things, the application of the ideas of the study of higher order efficiency, admissibility and minimaxity. In the matter of expansions, it is as important to have usable, explicit formulas as a rigorous proof that the expansions are valid in the sense of truly approximating a target quantity up to the claimed degree of accuracy.

Classical asymptotics is based on the notion of asymptotic distribution, often derived from the central limit theorem (see ►Central Limit Theorems), and usually the approximations are correct up to $O(n^{-1/2})$, where n is the sample size. Higher order asymptotics provides refinements based on asymptotic expansions of the distribution or density function of an estimator of a parameter. They are rooted in the Edgeworth theory, which is itself a refinement of the central limit theorem. The theory of higher order asymptotic is very much related with the corresponding to *Approximations to distributions* treated as well in this Encyclopedia.

When higher order asymptotic is correct up to $o(n^{-1/2})$, it is second order asymptotic. When further terms are picked up, so that the asymptotic is correct up to $o(n^{-1})$, it is third order asymptotic. In his pioneering papers, C. R. Rao coined the term second order efficiency for a concept that would now is called third order efficiency. The new terminology is essentially owing to Pfanzagl and Takeuchi.

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Cross References

►Approximations to Distributions

►Edgeworth Expansion

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