We discuss a generalization of Fisher's Inequality.

**Def:** Let \( m \in \mathbb{N} \), we say \( F \subseteq 2^{[m]} \) is \( L \)-intersecting if \( |A \cap B| \leq L \) for all \( A, B \in F \).

**Rmk:** Fisher's Ineq. just says that \( L = 1 \).

*Thrill Frankl - Wilson, 1981*

If \( F \) is an \( L \)-intersecting family in \( 2^{[n]} \), then \( |F| \leq \frac{n}{\binom{n-1}{L}} \).

**Rmk:** This is best possible, considering all subsets of \([n]\) of size at most \(L\) for \( n = 3\), \( 1 \leq L \leq 3 \implies |F| = \frac{1}{\binom{n-1}{L}} \).

**Proof:** Let \( F = \{ A_1, A_2, \ldots, A_m \} \) where \( |A_1| = \lambda_1 \leq \cdots \leq |A_m| \).

**Lemma:** If \( f_1, \ldots, f_m \) are linearly independent.

For each \( A_i \in F \), define a polynomial \( f_i(x) \) on \( \mathbb{R}^n \) by
\[
f_i(x) = \prod_{j \in A_i} (x_j - a_j).
\]

so \( f_i(x) \) is a polynomial with \( n \) variables and with degree \( \leq |A_i| \).

**Note:** \( f_i(A) = \prod_{j \in A_i} (|A_i| - 1) \), \( \forall A \subseteq [n] \).

For \( 1 \leq j \leq m \), \( f_j(X_j) = \prod_{i \neq j} (|A_i \cap A_j| - 1) = 0 \), since \( F \) is \( L \)-intersecting.

\[\exists (L, \delta) \text{ s.t. } |A_i \cap A_j| \leq \delta \times |A_i| \text{ where } |A_j| \leq |A_i| \implies |A_i \cap A_j| < |A_i| \).

By Lemma, we see \( f_1(x), \ldots, f_m(x) \) are linearly independent.

Next, we consider the dimension of the space containing such polynomials.

For \( x \in \mathbb{R}^n \), \( f_i(x) = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \ldots \), can view \( x_j \) as \( x_j \).

**Observation:** All vectors we consider are \( 0/1 \) vectors. Thus, we can define \( f_i(X) \) from \( f_i(x) \) by replacing \( x_j^k \) with \( X_j \).

\[\Rightarrow f_i(\mathbf{1}_j) = f_i(\mathbf{1}_j) = \delta, \quad j \leq \epsilon.\]

resulting in \( f_1(\mathbf{1}), \ldots, f_m(\mathbf{1}) \) are linearly independent.
We see that each \( f_i(x) \) is a linear combination of the monomials \( \prod x_i \) where \( I \in \mathcal{I}_n \) and \( |I| \leq |I'| \). And the number of such monomials is \( \frac{\binom{m}{k}}{k!} \), which is the dimension of the space containing \( f_1, \ldots, f_m \).

\[ |I| = m \leq \frac{\binom{m}{k}}{k!}. \]

**Thm 2.** Let \( p \) be a prime and \( L \leq \mathbb{Z}_p = \mathbb{F}_p \). Let \( I \leq \mathbb{Z}_p \) be s.t. \( |I| \) is a divisor of \( L \), \( |I| = 1 \pmod{p} \).

Then \( |I| \leq \frac{\binom{m}{k}}{k!} \).

\( F \) is \( L \)-intersecting. For Thm 1 in \( L \), we can find \( p \mid L \), \( L \equiv 0 \pmod{p} \), by Thm 2.

**Proof:** All operations are \( \pmod{p} \).

Define a polynomial \( f_i(x) \) over \( \mathbb{F}_p \) for each \( A_i \in F = \bigcap A_i \).

Then we have:

\[ f_i(I_{\mathbb{F}_p}) = \prod (x_i, x_i \in I_{\mathbb{F}_p} - 1) = 0 \text{ where } |A_i| \neq L. \]

\[ f_i(I_{\mathbb{F}_p}) = \prod (x_i, x_i \in A_i - 1) = 0. \text{ } \forall \mathbb{F}_p - 1. \]

\( f_1, f_2, \ldots, f_m \) are linearly independent over \( \mathbb{F}_p \).

Similarly, we replace each \( f_i(x) \) by \( f_i(x) \), where each factor \( x_i \) is replaced by \( x_i \). As we only consider \( \mathbb{F}_p \) vectors, this does not affect the above properties. So \( f_1, \ldots, f_m \) remains linearly independent, which we are generated by monomials \( \prod x_i \) with \( I \in \mathcal{I}_n \), \( |I| \leq |I'| \).

\[ |I| \leq m \leq \text{dimension} \leq \frac{\binom{m}{k}}{k!}. \]
Application:
We have constructed a graph on \( n = (p^3) \) vertices, which does not contain any clique or independent set of size \( k \).
\[ k \left( k-1; \binom{k}{2} \right) > \binom{k}{2} = \frac{k^2}{2} \]

We will improve the number of vertices to \( n \approx k^{2 \log \log k} \). 

**Thm 3 (Frankl-Wilson):** For any prime \( p \), there is a graph \( G \) on \( n = (p^3) \) vertices, such that the size \( k \) of maximum clique or independent set in \( G \) is at most \( \frac{\log (p^3)}{\log \log (p^3)} \).

Proof: \( G = (V, E) \) is defined as follows:
\[ V = \left\{ \binom{p^3}{2} \right\} \text{ and for } A, B \in V, \text{ if } \{A, B\} \in E \text{ iff } |A \cap B| = p-1 \text{ (mod } p) \]

1) Consider a maximum clique, say with vertices \( A_1, \ldots, A_k \).
\[ |A_i \cap A_j| = p-1 \text{ (mod } p) \quad i \neq j \]
\[ |A_i| = p^3 - 1 = p^3 - 1 \text{ (mod } p) \]

We can use Thm 2 by letting \( L = \left\{ 0, 1, \ldots, p-2 \right\} \subseteq \mathbb{Z}_p \), so \( k \leq \frac{p^3}{\log (p^3)} \).

2) Consider a maximum stable set, say \( B_1, \ldots, B_k \).
\[ |B_i \cup B_j| = p-1 \text{ (mod } p) \quad i \neq j \]
\[ |B_i| = p^3 - 1 \]

So \( |B_i \cup B_j| \in L = \left\{ p-1, 2p-1, \ldots, (p-3)p + 1 \right\} \), \( |L| = p+1 \) and \( B_1, \ldots, B_k \) are \( L \)-intersecting.

By Thm 1, \[ k \leq \frac{p^3}{\log (p^3)} \]

\[ n = \left( \frac{p^3}{p^3} \right) \leq p \Theta(p^3) \]
\[ k = \frac{p^3}{\log (p^3)} \]
\[ \log k = \log p \log \log p \leq \log p \log \log p \leq \log p \]
\[ p \geq \frac{\log k}{\log \log p} \]
\[ p = \frac{\log k}{\log \log p} \approx \frac{\log k}{\log \log p} \]
\[ n \approx p \Theta(p^3) = (p^3) \leq k^p \leq k^{\log \log p} \]

**Thm 5:** \( p(k, k) \geq p \)
Def: Given a set \( S \subseteq \mathbb{R}^n \), (bounded) the diameter of \( S \) is denoted as 
\[
\text{Diam}(S) = \sup \{ d(x, y) : x, y \in S \}
\]

Euclidean distance between \( x \) and \( y \) in \( \mathbb{R}^n \).

Borsuk's Conjecture: Can every bounded \( S \subseteq \mathbb{R}^d \) be partitioned into \( d+1 \) sets of strictly smaller diameter?

Known: \( S = \text{Sphere} \), \( S = \text{a smooth convex body} \), \( d = 3 \).

In 1993, Kahn-Kalai disproved this conjecture.

Lemma: For prime \( p \), there exists a set of \( \frac{1}{2} \binom{4p}{p} \) vectors in \( \mathbb{F} = \mathbb{F}_p, 13^{4p} \), such that every subset of size \( 2 \binom{4p}{p} \) vectors contains an orthogonal (balanced) pair of vectors.

Proof: Let \( Q = \{ I \in \binom{4p}{p} : I \in \mathbb{F} \} \).

For \( A \subseteq Q \), define a vector \( \vec{v}^A \in \mathbb{F}^{4p} \) by
\[
\vec{v}^A_i = \frac{1}{\sqrt{2}} \chi_{I, \mathbb{F}}, \quad x \in I.
\]

Let \( \mathcal{B} = \{ \vec{v}^I : I \in Q \} \).

Compute:
\[
\vec{v}^I \cdot \vec{v}^J = |I\cap J| - |I\Delta J| = |I\cap J| + |I\Delta J| = 4p - 2|I\cap J|
\]

so \( \vec{v}^I \perp \vec{v}^J \iff |I\cap J| = 2p = |I| + |J| - 2|I\cap J| = 4p - 2|I\cap J| \Rightarrow |I\cap J| = p \).

Note that \( |I| + |J| = 2p \Rightarrow 1 \leq |I\cap J| \leq 2p - 1 \). \( \forall I \cap J \), it holds \( |I\cap J| \neq 0 \) (mod \( p \)).

Claim: For any subset \( G \subseteq \mathcal{B} \) without orthogonal pairs, the \( |G| \leq \frac{p}{k^2} \binom{4p}{p} \) \( (p) \).

Proof: Let \( A \in G \), \( |A| = 2p \equiv 0 \) (mod \( p \)).

Taking \( L = \{ i, j, \ldots, p+1 \} \leq 2p \).

By Thm. 2, \( |G| \leq \frac{p}{k^2} \binom{4p}{p} \leq 2 \frac{p^{4p}}{p^{p-1}} \).

Claim2 (Exercise)

Thm. 4: For \( d \) sufficiently large, there exists a bounded \( S \subseteq \mathbb{R}^d \) (in fact a finite set) so that any partition of \( S \) into \( d+1 \) contains
a part of the same character.
As $1.1^{11} >> d + 1$, this disproves Borsuk's conj.

Proof: Let $G$ be from the Lemma.

(Def: A tensor product of vector $\vec{v} \in \mathbb{R}^n$ is
\[ \vec{w} = \vec{v} \otimes \vec{v} \in \mathbb{R}^{n^2} \] by $W_{ij} = v_i \cdot v_j$, $1 \leq i,j \leq n$.)

Let $X = \{ \vec{x} \in \mathbb{R}^n : \vec{x} \in F \}$ be $\mathbb{R}^{n^2}$, $n = 4p$.

\cdot $\vec{w} \in F \cap F'$.
\[ \|\vec{w}\| = n^2 \Rightarrow \|\vec{w}\| = n. \]

\cdot $\vec{w}, \vec{w}' \in \mathbb{R}^n$, say $\vec{w} = \vec{v} \otimes \vec{v}$, $\vec{w}' = \vec{v} \otimes \vec{v}'$

\[ \vec{w} \cdot \vec{w}' = \sum\overline{W_{ij} W_{ij}'} = \sum\overline{V_{ij} V_{ij}'} = (\overline{V \cdot V'})^2 \]

\cdot $\vec{w} \perp \vec{w}'$ if $\vec{w} \cdot \vec{w}' = 0$

\[ \|\vec{w} - \vec{w}'\|^2 = \|\vec{w}\|^2 + \|\vec{w}'\|^2 - 2 \vec{w} \cdot \vec{w}' = 2n^2 - 2 (\overline{V \cdot V'})^2 \leq 2n^2. \]

\[ \Rightarrow \text{Diam}(X) = \{\mathbb{R}\} \]

By the lemma, any subset of $2(\frac{4^p}{p-1})$ vectors in $F$, contains
an orthogonal pair. Thus, any subset of $2(\frac{4^p}{p-1})$ vectors in $X$
contains a pair $\vec{w}, \vec{w}'$, of max distance = $\mathbb{R}$. Therefore, if we want to decrease the diameter, we must partition
$X$ into sets of size less than $2(\frac{4^p}{p-1})$, and so the number of parts
\[ \frac{\|X\|}{2(\frac{4^p}{p-1})} = \frac{\frac{1}{2}(\frac{4^p-1}{p})}{2^{(\frac{4^p}{p-1})}} = \frac{1}{2} \cdot \frac{(4^p)\cdots (4^{p+1})}{2p(2p+1)\cdots \text{p}} \geq \frac{1}{4} \cdot \frac{3}{2} \Rightarrow 2 \cdot (\frac{3}{2}) \Rightarrow \text{111} \]

$X \leq Rd = \mathbb{R^n}$, $d = n^2 = (4p)^2 = 16p^2$. #
Recall: Sperner's Thm: For any antichain $F \subseteq 2^{[n]}$ (i.e., $\forall A, B \in F, A \cap B = \emptyset$) then we have $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

In fact, we proved a stronger result:

LYM-Inequality: For any antichain $F \subseteq 2^{[n]}$, \[ \sum_{A \in F} \frac{1}{|A|} \leq 1 \]

Today, we study an even stronger result, namely, the Bollobás's Thm (extremal set theory)

Bollobás's Thm: Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be two sequences of sets such that:

- $A_i \cap B_j \neq \emptyset$, $\forall i, j$
- $A_i \cap B_i = \emptyset$, $\forall i$

Then \[ \sum_{i=1}^{m} \frac{1}{|A_i \cap B_i|} \leq 1 \], where $a_i = |A_i|$ and $b_i = |B_i|$

Exercise: Prove that Bollobás's Thm can imply LYM-Inequality. ($b_i = a_i$)

Proof: Let $x = \bigcup_{i=1}^{n} (A_i \cup B_i)$, we prove by induction on $|x| = n$.

When $n = 1$, clearly \[ \emptyset \]

so we assume it holds for $|x| = m$. For each $x \in x$, define \[ I_x = \{ s \subseteq m : x \in A_s \} \]

Define \[ F_x = \{ A_s : s \in I_x \} \cup \{ B_s - x_j : s \in I_x \} \]

Note that any set of $F_x$ doesn't contain $x$, so $F_x$ has less than $n$ elements.

Hence we apply induction hypothesis for each $F_x$. (Check $F_x$ satisfies the condition)

\[ \sum_{x \in x} \frac{1}{|A_s|} \leq 1 \]

We summing up the above inequalities for all $x \in x$

\[ \sum_{x \in x} \sum_{s \in I_x} \frac{1}{|A_s|} \leq n \]

For each $i$, it contributes either $0$ or $\frac{1}{|A_i|}$ or $\frac{1}{|A_i \cap B_i|}$ to each $x$.\[ \sum_{x \in x} \]
The term $\frac{ax + by}{\overline{ax}}$ corresponds to points $x \in A \cup B \cdot y \cdot x \in B \cdot$, thus this term appears exactly $(n = a_x - b_y)$ times.

While, the term $\frac{ax + by}{\overline{ax}}$ corresponds to points $x \in A \cap y \cdot x \in B \cdot$, thus this term appears exactly $b_x$ times.

$\Rightarrow \frac{1}{\sum a_x} \left[ (n - a_x - b_y) \left( \frac{ax + by}{\overline{ax}} \right) + b_x \left( \frac{ax + by}{\overline{ax}} \right) \right] \leq n.

Since $\frac{k}{b} = \frac{k - b}{b}$, we get $\frac{1}{\overline{ax}} \Rightarrow \frac{ax + by}{\overline{ax}} = \frac{a_x + b_y}{b_x}$.

Plugging in, $\sum n \cdot \left[ (n - a_x - b_y) \left( \frac{ax + by}{\overline{ax}} \right) + \frac{a_x + b_y}{b_x} \right] \leq n$.

$\Rightarrow \sum \frac{n - a_x - b_y}{\overline{ax}} \leq n$.

#

**Def:** Let $F$ be a field, a set $A \subseteq F^n$ is in general position, if any $n$ vectors in $A$ are linearly independent over $F$.

E.g. for $a \in F$, define $\text{span}(a) = \{a^0, a^1, a^2, \ldots, a^n\} = F^n$ (moment curve)

Then $\text{span}(a) \cap A = F \{a \in F\}$ is in general position. $\{4|\text{basis}\} + 3\times$

Next, we use the so-called "general position" argument to prove a version of Bollobás's Thm, which is weaker than the previous one. But, on the other hand, the condition can be generalized to $A_i \cap B_j \neq \emptyset$ for $i < j$.

**Bollobás's Thm (the skew version)**

Let $A_1, \ldots, A_m$ be sets of size $1$ and $B_1, \ldots, B_m$ be sets of size $s$, such that:

- $A_i \cap B_j \neq \emptyset$, $i < j$
- $A_i \cap B_i = \emptyset$, $i < j$

Then, $m \leq \binom{1 + t}{s}$. 
Proof (by Lovász): Let \( X = \mathcal{Y}(A_1 \cup V B_i) \)

Take a set \( V \subseteq \mathbb{R}^{r+1} \) of vectors \( \bar{V} = (v_0, v, \ldots, v_r) \) such that
- \( V \) is in general position
- \( |V| = 1 \times 1 \).

Identify the elements of \( X \) with vectors in \( V \). \( X \leftrightarrow V \)
Hence, we will view \( A_i \) as a subset in \( V \) containing \( r \) vectors and \( B_j \) as a subset in \( V \) containing \( s \) vectors.

For each \( B_j \), define \( f_j (\bar{x}) = \prod_{v \in B_j} \langle \bar{v}, \bar{x} \rangle = \prod_{v \in B_j} (v_0 x + \ldots + v_{r-1} x_r) \).

For \( x \in \mathbb{R}^{r+1} \), note that \( f_j (x) = 0 \iff \langle \bar{v}, \bar{x} \rangle = 0 \) for some \( v \in B_j \).

Consider the subspace \( \text{span} A_i \), which is spanned by the \( r \) vector in \( A_i \).
Since \( A_i \subseteq V \subseteq \mathbb{R}^{r+1} \) and \( V \) is in general position, we see that all \( r \) vectors in \( A_i \) are linearly independent and thus \( \dim(\text{span} A_i) = r \).

So, \( (\text{span} A_i)^\perp \) has dimension 1. Choose \( a_i \in (\text{span} A_i)^\perp \) for \( i = 1, \ldots, m \).
Then for each \( \bar{v} \in V \), \( \langle \bar{v}, \bar{a} \rangle = 0 \iff \bar{v} \in \text{span} A_i \).
\( \forall \bar{v} \in A_i \)
\( (\text{dim.} \bar{v} + A_i, f_j / VA_i \) has \( r+1 \) vectors in \( V \), which must be linearly indep.
\( \land \text{contradicts to } \bar{v} \in \text{span} A_i \).

Combining \( \odot \) and \( \oplus \), \( f_j (\bar{a}) = \prod_{v \in B_j} \langle \bar{v}, \bar{a} \rangle = 0 \iff A_i \cap B_j \neq \emptyset \).
\( \Rightarrow \sum_{j=1}^{m} f_j (\bar{a}) = 0, \forall i \in \mathbb{N} \)
\( f_j (\bar{a}) \neq 0 \) (since \( A_j \cap B_j = \emptyset \)), \( \forall j \).

This shows that \( f_1, \ldots, f_m \) are linearly indep. \( (f_j (\bar{a})) = \left( \begin{array}{c} \vdots \ 0 \ \vdots \end{array} \right) \)

Next, we give an upper bound on the dimension of the space containing \( f_1, \ldots, f_m \).
Recall: \( f_j (x) = \prod_{v \in B_j} (v_0 x + \ldots + v_{r-1} x_r) \). It is homogeneous with degree \( s = |B_j| \) and \( r+1 \) variables \( (x_0, x_1, \ldots, x_r) \). So this polynomial space can be generated by all monomials of follows:
\( x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r}, \text{ where } i_0 + i_1 + \cdots + i_r = s, \ i_r > 0 \).
\( \Rightarrow (r+2) \choose (r+1) = (r+1) \choose r \).

There are \( (r+1) \choose r \) many solutions !. So \( m \leq \text{the dimension} = (r+1) \).
EVERYTHING LET'S HEARTBEAT

Subspace version: $V_1, \ldots, V_m$ be subspaces of dimension $r \leq \ldots \leq \ldots \leq s$.

\[ V_i \cap V_j = \emptyset, \quad i \neq j \Rightarrow m \leq \binom{r+s}{r}. \]

(29th)

12.21

Covering by complete bipartite subgraphs.

The following question was motivated by telephone communication problem.

Q: Determine the minimum $m = m(n)$ s.t. the edge set $E(K_n)$ can be expressed as a disjoint union of edge sets of $m$ complete bipartite subgraphs of $K_n$.

\[ K_5 = \star + K_{1,4} + K_{1,3} + K_{1,2} + K_{1,1}. \]

Fact: $m(n) \leq n-1$.

Pf: Because we can express $E(K_n)$ as a disjoint of $n-1$ stars. #

Rmk: $K_5 = \star + \star + \star + \cdots + \star$.

We point out that there exist other partitions of $E(K_n)$, using $n-1$ complete bipartite subgraph, which is not isomorphic to the star-decomposition.
Fact 1: For any n x n matrices $M_1$, $M_2$, \( \text{rank}(M_1 + M_2) \leq \text{rank}(M_1) + \text{rank}(M_2) \) on n vertices.

**Def:** The adjacency matrix of a graph $H$ is an $n \times n$ matrix $A = (a_{ij})_{n \times n}$ such that $a_{ij} = 1$, $i \neq j \in E(H)$, and $a_{ii} = 0$ (so $A$ is symmetric).

**Thm (Graham-Pollak):** $m(n) \geq n - 1$.

**Proof:** Suppose the complete bipartite graphs $B_1, B_2, \ldots, B_m$ disjointly cover all edges of $K_n$, i.e., $E(K_n) = E(B_1) \cup \cdots \cup E(B_m)$.

Let $X_i$ and $Y_i$ be the color classes of $B_i$, i.e., all edges of $B_i$ go between $X_i$ and $Y_i$. For each $B_i$, we define an $n \times n$ matrix $A_i = (a_{ij}^{(B_i)})_{n \times n}$ by $a_{ij}^{(B_i)} = \{ 1 \quad \text{if } i \in X_k \cap Y_k \\
0 \quad \text{otherwise} \}$.

\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

**Claim:** $\text{rank}(A_i) = 1$, $\forall i \leq m$.

**Proof:** Because $B_i = (X_i, Y_i)$ is complete, the $i^{th}$ row (for $i \in X_i$) are identical.

Let $A = A_1 + A_2 + \cdots + A_m$. As each $ij \in E(K_n)$ belongs to exactly one of the graphs $B_i$, we have $a_{ij}^{(B_1)} = 1$, $a_{ij}^{(B_2)} = 0$ or $a_{ij}^{(B_i)} = 0$, $a_{ij}^{(B_i)} = 1$.

$\Rightarrow A + A^T = J_n - I_n$, where $J_n = (1)_{n \times n} = A(K_n)$.

Note that $\text{rank}(A) \leq \sum_{i=1}^m \text{rank}(A_i) = m$.

It suffices to prove: $\text{rank}(A) \geq n - 1$. Suppose a contradiction that $\text{rank}(A) < n - 1$. Let $A'$ be an $(n+1) \times n$ matrix obtained from $A$ by adding an extra row $(1, 1, \ldots, 1)$.

$\Rightarrow \text{rank}(A') \leq \text{rank}(A) + 1 < n - 1 + 1 = n$. Therefore, $\text{rank}(A') = n - 1$.

So, $\text{rank}(A') = n - 1$. 

Then there exists non-zero vector $\mathbf{x} \in \mathbb{R}^n$ s.t. $A^T \mathbf{x} = \mathbf{0}$.
$\implies A \mathbf{x} = 0$ & $\sum_i x_i = 0$.

Consider $\mathbf{x}^T (A + A^T) \mathbf{x} = \mathbf{x}^T (I_n - I_n) \mathbf{x}$
$\implies 0 = \mathbf{x}^T (A \mathbf{x}) + (A^T \mathbf{x}^T) = \mathbf{x}^T (I_n \mathbf{x}) - \mathbf{x}^T I_n \mathbf{x} = 0 - \frac{2}{E} x_i^2 < 0$. contradiction

Thus, $\text{rank}(A) > n - 1$.
$\implies \min(\text{rank}) > n - 1$.

---

**Finite Projective Plane (FPP)**

**Def:** Let $X$ be a finite set and $\mathcal{L} \subseteq 2^X$ be a family. The pair $(X, \mathcal{L})$ is called a finite projective plane (FPP for short).

if it satisfies three axioms:

1. **(P0)** There exists a 4-set $T \subseteq X$ s.t. $|T \cap L| \leq 2$ for all $L \in \mathcal{L}$.
2. **(P1)** $\forall L_1, L_2 \in \mathcal{L}$ has $|L_1 \cap L_2| = 1$
3. **(P2)** $\forall x_1, x_2 \in X$, there exists exactly one subset $L \in \mathcal{L}$ with $\{x_1, x_2\} \subseteq L$.

We call elements of $X$ as points and the sets in $\mathcal{L}$ as lines.

$\implies$ **(P1):** 两条线交于一点  
**(P2):** 两点确定一条直线

**PO?**

**(not-interesting)**

So we explain for (P0) - (P2).

**(P1)** says: $\forall$ 2 lines intersect at exactly one point

(In geometry, parallel lines do not!)

**(P2)** says: $\forall$ 2 points $a, b$ determine a line, denoted by $ab$.

**(P0)** is used to exclude some not-interesting cases.
The Fano plane (the smallest FPP)

\[ \begin{align*}
\{1, 2, 3\} & \quad \{3, 4, 5\} & \{5, 6, 1\} \\
\{1, 4, 7\} & \quad \{3, 6, 7\} & \{2, 5, 7\} \\
\{2, 4, 6\} &
\end{align*} \]

\( \times \) 7 points & 7 lines, each line with 3 points.

**Prop 1.** Let \((X, L)\) be a finite projective plane, then for any 2 lines \(L, L' \in L\), \(|L | = |L'|\).

**Proof:** Claim: \( \exists x \neq x' \) with \( x \not\in L \cup L' \)

**(I):** Let \( F \leq X \) be from \((P_0)\). Then \(|F \cap L | \leq 2 \) & \(|F \cap L' | \leq 2 \).

If \( F \not\subset L \cup L' \), then we are done, so \( F \not\subset L \cup L' \) with \(|F \cap L | = |F \cap L' | = 2 \).

Let \( F \cap L = \{a, b\} \), \( F \cap L' = \{c, d\} \).

Let \( L_1 = \overline{ac} \), \( L_2 = \overline{bd} \). Let \( z \in L_1 \cap L_2 \) be the unique point.

If \( z \not\in L \), \( z \not\in L' \), then again we are done.

So we assume \( z \in L \). \( \Rightarrow z \in L \cup L' \). But \( a \not\in L \cup L' \).

By \((P_1)\), \( z = a \in L_2 \), \( b \in L \). \( \Rightarrow b \in L \ \wedge F \not\subset \). a contradiction \( \not\subset \) \((P_0)\)!

Now fix \( x \not\in L \cup L' \). We define a mapping \( \varphi : L \to L' \) as follows: for \( y \in L \), let \( \varphi(y) \in L' \) be the unique point in \( L' \setminus L \).

Next we show \( \varphi \) is bijective.

- **\( \varphi \) is injective:** If \( \Rightarrow a_1, a_2 \in L \) st. \( \varphi(a_1) = \varphi(a_2) \)

then \( a_1, a_2, \varphi(a_1), x \) are in the line \( \varphi(a_1) \cap \varphi(a_2) = \overline{a_1a_2} \).

But \( \varphi(a_1), \varphi(a_2) \in \varphi(a_1) \setminus L \), a contradiction to \((P_2)\).

- **\( \varphi \) is surjective:** For any \( b \in L' \), let \( a \) be the unique point in \( \overline{b} \cap L \). \( a, b, x \) are in the line \( \overline{a} = \overline{b} \).

\( \Rightarrow b \in L \cap L' \Rightarrow \varphi(a) = b \).

\( \Rightarrow \varphi \) is bijective. \( \Rightarrow |L | = |L'| \). #
**Def:** Let \((X, \mathcal{L})\) be a finite projective plane. The order of \((X, \mathcal{L})\) is the number \(141 - 1\) for \(X \in \mathcal{L} \subseteq \mathcal{L}.

**Prop. 2:** Let \((X, \mathcal{L})\) be a FPP of order \(n\). Then

(i) exactly \(n+1\) lines pass through each point \(x \in X\).

(ii) \(|X| = n^2 + n + 1\).

(iii) \(|\mathcal{L}| = n^2 + n + 1\).

**Proof:**

(i) Consider \(X \times X\). Let \(F\) be the 4-set satisfying (P0).

\[F = \{a, b, c, d\}\]  
Let \(a, b, c, d \in \mathcal{L}\) be distinct from \(x\).

- Then, at least one of lines \(ab\), \(ac\) doesn't contain \(x\). (otherwise \(a, b, c, x\) are in the same line, contradicting (P0).)

- There exists a line \(L\) with \(x \notin L\). Let \(L = \{x_0, x_1, \ldots, x_n\}\); then \(x_0\) defines \(n+1\) lines.

- On the other hand, any line containing \(x\) must intersect \(L\) at some point; say \(x_c\). Therefore, there are exactly \(n+1\) lines containing \(x\).

(ii) Choose some line \(L = \{x_0, x_1, \ldots, x_n\} \in \mathcal{L}\) and a point \(a \in X\) with \(a \notin L\). Let \(L_i = \overline{ax}\) for \(i = 0, 1, \ldots, n\).

By (P1), any two lines \(L_i, L_j\) intersect at a single point, that is a. So \(|L_0 L_1 U \ldots U L_n| = n(n+1) = n^2 + n + 1\).

It remains to show that any \(x \in X - \{x_0\}\) must belong to \(L_0 L_1 U \ldots U L_n\).

By (P1), line \(L_i\) must intersect \(L\) at some point \(x_i\), then \(\overline{ax} = \overline{ax_i} = L_i \Rightarrow x \in L_i\).

\[\Rightarrow x = L_0 L_1 U \ldots U L_n\.

\[\Rightarrow |X| = n^2 + n + 1.\]

(iii) Exercise.
**Def:** The incidence graph of a FPP $(X, L)$ is a bipartite graph $G$ with parts $X$ and $L$, where $x \in X$ is adjacent to $L$ if $x \in L$.

![Incidence graph diagram]

From this we can prove that $|X| = |L|$. (Note: $|X| = |L| = 12$ in this case.)

**Def:** The dual $(L, X)$ of a FPP $(X, L)$ is obtained by taking the incidence graph $G$ of $(X, L)$ and interpreting its points in $(X, L)$ as the lines in the new system and the lines in $(X, L)$ as the points in the new system, swapping the roles of "points" and "lines".

**Prop:** If $x \in X$, then $L_x = \{ L \in L : x \in L \}$.

$X = \{ L_x : x \in X \}$.

**Prop 3:** The dual $(L, X)$ is also a FPP.

**Proof:** We point out that for $i = 1, 2$,

$(P_i)$ for $(X, L)$ gives rise to $(P_i^*)$ for $(L, X)$.

$(P_1) \Rightarrow (P_2)^*$: $\forall L, L_2 \in L$, $\exists ! L_x \in X$, s.t. $L \in L_x \Leftrightarrow x \in L \land L_2 \Leftrightarrow x \in L \land L_2$.

$(P_2) \Rightarrow (P_1)^*$: $\forall L_2, L_2 \in L$, $\exists ! L_x \in X$, s.t. $L \in L_x \Leftrightarrow x \in L \land L_2 \Leftrightarrow x \in L \land L_2$.

$(P_0)^*$: It suffices to show that $(P_0)$ holds for $(L, X)$, that is

we need to find 4 lines $L_1, L_2, L_3, L_4$, s.t. $\forall 3$ of them have a common point!

Let $F = \{ a, b, c, d \}$ be the 4-set satisfying $(P_0)$ for $(X, L)$.

Note that for such $F$, $|F \land L| \leq 2$, $\forall L \in L$.

Since any 3 points of $F$ do not lie on a line, we can define 4 distinct lines as follows:

$L_1 = ab$, $L_2 = ac$, $L_3 = ad$, $L_4 = bc$.

By Property of $F$, for any 3 lines of $\{ L_1, L_2, L_3, L_4 \}$ (by symmetry, say $L_1, L_2, L_3$),

we see $L_1 \land L_3 = \{ a \}$, $L_2 \land L_3 = \{ b \} \Rightarrow L_1 \land L_2 \land L_3 = \emptyset$. This proves $\Box$. #
Think: A finite projective plane of order $n$ exists whenever a field with $n$ elements exists.

And we know a field with $n$ elements exists if $n = p^k$ for a prime $p$.

Q: $n = p^k$, exists? NO known.

An Application of FPP:

Recall: A $4$-free graph $G$ on $m$ vertices has $e(G) = \frac{m}{2} \left( 1 + \frac{1}{m-3} \right)$.

Think: For infinitely many integers $m$, there exists a $4$-free graph on $m$ vertices and with at least $0.35 m^2$.

Proof: Take a FPP $(X, L)$ with order $n$ and take its incidence graph $G$.

$G$ has $m = |X| + |L| = 2(n^2 + n + 1)$ vertices and $e(G) = \frac{n}{3} \cdot d_G(x) = \frac{1}{3} (n+1)(n^2 + n + 1)$.

$e(G) = (n+1)(n^2 + n + 1) > (n^2 + n + 1)^{\frac{3}{2}} = (n^3 + n^2 + n + 1)^{\frac{3}{2}} \geq 0.35 m^2$.

why does such $G$ have NO $C_4$?

If existing, then in the language of the FPP, it says there exist 2 points $x, x'$ and 2 lines $l_1, l_2$ such $x \in l_1$, $x' \in l_2$ contradiction to $(P1)$ and $(P2)$.

#