Exercises on Chapter 1

1.1 Prove the inequalities (a) \( \sim \) (c) in Theorem 1.2.2.

1.2 Prove Theorem 1.2.3 and Theorem 1.2.5.

1.3 Prove that \( \gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( \gamma \leq \alpha(G) \) for a connected graph \( G \) of order \( n \geq 2 \).

1.4 Count the number of vertices of a complete complete \( k \)-ary tree with height \( h \).

1.5 Give an embedding of a complete binary tree with height two into a cube \( Q_3 \) with no slowdown.

1.6 Prove \( d(G) \leq \frac{3n-\delta-3}{4+1} \) for a connected graph \( G \) with order \( n \). (Soares [4])

1.7 Prove Theorem 1.4.4 by a self-contained language. (Ore [3])

1.8 Prove that the average distance of an undirected cycle \( C_n \) is equal to
\[
\mu(C_n) = \begin{cases} 
\frac{n+1}{n}, & \text{if } n \text{ is odd;} \\
\frac{n^2}{4(n-1)}, & \text{if } n \text{ is even.}
\end{cases}
\]

1.9 Prove Theorem 1.4.5. Can you improve the two lower bounds?

1.10 Let \( G \) be a undirected connected graph and for a vertex \( x \) in \( G \) let
\[
\sigma_x(G) = \sum_{y\in V\setminus\{x\}} d(G; x, y).
\]
Prove that \( \sigma_x(G) \leq \left\lfloor \frac{1}{4} v^2 \right\rfloor \) for any \( x \in V \). (Plesnik [2])

1.11 Read the reference [1] and give a proof of Theorem 1.4.6.

1.12 Prove that if \( G \) is a \( k \)-connected graph of order \( n \), then number of pairs of distinct vertices at distance at most two in \( G \) at least \( \min\{2kn, n(n-1)\} \).

References


[2] Plesnik, J., On the sum of all distances in a graph or digraph. Journal of Graph Theory, 8 (1984), 1-21
